Lecture 29
Identification of uncertain systems

– Purpose
– Discrete-time uncertain system
– Worst-case simulation
– Case with no uncertainty
– General case
– Examples

Purpose of identification

The purpose of identification is to build a model consistent with past observation, based on

– a given horizon of analysis (time horizon for example)
– some assumptions about the model (*e.g.*, linearity, stationarity, etc)

In this lecture: we will identify uncertain, time-varying models from time-series

Example: Interval ARMA processes

An ARMA-like process:

$$y(k+N) + \mathbf{a}_1(k)y(k+N-1) + \ldots + \mathbf{a}_N(k)y(k) = \mathbf{b}_1(k)u(k+N-1) + \ldots + \mathbf{b}_N(k)u(k)$$

where $u(k)$ is (known) input, $y(k)$ is (measured) output

Define time-varying, "coefficient vector"

$$\mathbf{c}(k) := \begin{bmatrix} \mathbf{a}(k) \\ \mathbf{b}(k) \end{bmatrix}$$

so recurrence relation above writes

$$\omega(k)^T \mathbf{c}(k) = \gamma(k), \quad k = 0, \ldots, T$$

where $\omega(k)$ is a vector and $\gamma(k)$ a scalar:

$$\omega(k) := \begin{bmatrix} y(k+N-1) \\ \ldots \\ y(k) \\ -u(k+N-1) \\ \ldots \\ u(k) \end{bmatrix}, \quad \gamma(k) := y(k+N)$$

Purpose

For each k, recurrence equation

$$\omega(k)^T \mathbf{c}(k) = \gamma(k)$$

defines a hyperplane in space of coefficients $\mathbf{R}^{2N}$

Since the coefficient vector $\mathbf{c}$ is time-varying, there is an infinite number of possible choices for $\mathbf{c}(k)$ at each k, given the signals y, u

Aim: find the "smallest set" of $\mathbf{c}(k)$ that make the equations true

Ellipsoidal approach

Problem: minimize the "size" of an ellipsoid $\mathcal{E}$ such that:

for every k,

$$\omega(k)^T \mathbf{c}(k) = \gamma(k)$$

holds for some $\mathbf{c}(k) \in \mathcal{E}$

– Size of optimal ellipsoid is a measure of non-stationarity of signal

– Optimal ellipsoid is a singleton iff signal generated by stationary system of order N

Geometric interpretation

Parametrize ellipsoid in coefficient space ($\mathbf{R}^{2N}$) as

$$\mathcal{E}(P, c) = \{\xi \mid P \succeq (\xi - c)(\xi - c)^T\}$$

where $c \in \mathbf{R}^{2N}$ is the center and $P \in \mathbf{R}^{2N \times 2N}$ defines the shape

Measure size of $\mathcal{E}$ by $\mathbf{Tr}\, P$

Geometrically, we seek the "smallest" ellipsoid that intersects a given number of hyperplanes defined by

$$\omega(k)^T \mathbf{c} = \gamma(k), \quad k = 0, \ldots, T$$

SDP solution

Fact: (try to prove it!) Ellipsoid $\mathcal{E}(P, c)$ intersects hyperplane $\{\xi \mid \omega^T \xi = \gamma\}$ iff

$$\omega^T P \omega \geq (\omega^T c - \gamma)^2$$

hence, the optimal ellipsoid is given by the SDP

$$\begin{array}{ll} \text{minimize } \textbf{Tr}\, P & \text{subject to} \\ & P \succeq 0, \\ & \omega(k)^T P \omega(k) \geq (\omega(k)^T c - \gamma(k))^2, \\ & 0 \leq k \leq T \end{array}$$

Can be solved in time polynomial in N (the order of the system) and T (the horizon of analysis)

In practice: solve problem for several orders and choose the "best"

Affine uncertainty

Assume now system is given by state equation

$$x(k+1) = \mathbf{A}(k)x(k) + \mathbf{b}(k), \quad k = 0, 1, 2, \ldots$$

where

$$\forall\, k, \quad \mathbf{M}(k) := [\mathbf{A}(k)\ \mathbf{b}(k)] \in \mathcal{S}$$

and the set $\mathcal{S}$ is an affine subspace, given by

$$\mathcal{S} = \left\{ = [\mathbf{A}\ \mathbf{b}] = [A_0\ b_0] + \sum_{i=1}^{l} \mathbf{c}_i [A_i\ b_i] \;\middle|\; \mathbf{c} \in \mathbf{R}^l \right\},$$

where $[A_i\ b_i]$'s are given matrices, and $\mathbf{c}$ contains the "parameters" of the uncertain system.

SDP solution

We can rewrite state equation at time k as

$$\Omega(k)^T \mathbf{c}(k) = \Gamma(k), \quad k = 0, \ldots, T$$

where Ω is now a matrix and Γ is a vector:

$$\Omega(k) = \left[\, A_1 x_k + b_1 \; \ldots \; A_l x_k + b_l \, \right]^T,$$
$$\Gamma(k) = x_{k+1} - A_0 x_k - b_0$$

Fact: (try to prove it!) Ellipsoid $\mathcal{E}(P, c)$ intersects affine subspace $\{\xi \mid \Omega^T \xi = \Gamma\}$ iff

$$\Omega^T P \Omega \succeq (\Omega^T c - \Gamma)(\Omega^T c - \Gamma)^T$$

hence, the optimal ellipsoid is given by the SDP

minimize **Tr** P subject to
$P \succeq 0$,
$\Omega(k)^T P \Omega(k) \succeq (\Omega(k)^T c - \Gamma(k))(\Omega(k)^T c - \Gamma(k))^T$,
$0 \leq k \leq T$

Recursive Case

Previous approaches have complexity that grows with time horizon T

How can we do if the horizon gets ver large?

Assume we know an ellipsoid of confidence $\mathcal{E}$ from a given time series over $[0 \; T]$

Let us add a new measurement at time $T + 1$, resulting in a new linear constraint on coefficients

$$\Omega(T + 1)^T \mathbf{c}(T + 1) = \Gamma(T + 1) \qquad (1)$$

Seek an ellipsoid of minimal size $\mathcal{E}_+$, such that

1. $\mathcal{E}_+$ contains $\mathcal{E}$;
2. $\mathcal{E}_+$ intersects the affine set defined by the relation (1).

Interpretation

The condition $\mathcal{E}_+$ contains $\mathcal{E}$ ensures all past data (from time 0 to T) is consistent with the new uncertain model implied by $\mathcal{E}_+$

Second condition, *i.e.* new ellipsoid $\mathcal{E}_+$ intersects new hyperplane, ensures that the new data is also consistent

In effect, we summarize all information extracted from past data in one ellipsoid (as opposed to many hyperplanes)

So, even if we solve above problem exactly, approach is suboptimal!

SDP solution

Using Cholesky factor of current shape matrix P, describe current ellipsoid as

$$\mathcal{E}(P,c) = \{c + Eu \mid \|u\|_2 \leq 1\}$$

(E related to P by $P = EE^T$)

We want P_+, c_+ to be such that

$$(\xi - c_+)^T P_+^{-1} (\xi - c_+) \leq 1 \text{ whenever } \xi \in \mathcal{E}$$

Using Lagrange relaxation (exact in this case!), get the following necessary and sufficient condition:

there exists $\tau \geq 0$ such that

$$\begin{bmatrix} E & c_+ - c \end{bmatrix}^T P_+^{-1} \begin{bmatrix} E & c_+ - c \end{bmatrix} \preceq \begin{bmatrix} \tau I & 0 \\ 0 & 1-\tau \end{bmatrix}$$

Readily transformed into an LMI in P_+, c_+!

SDP solution: summary

An ellipsoid of confidence can be recursively computed by solved the SDP

$$\begin{aligned}&\text{minimize } \mathbf{Tr}\, P \text{ subject to}\\ &\begin{bmatrix} P_+ & E & c_+ - c \\ E^T & \tau I & 0 \\ (c_+ - c)^T & 0 & 1-\tau \end{bmatrix} \succeq 0,\\ &\Omega_+^T P \Omega_+ \succeq (\Omega_+^T c - \Gamma_+)(\Omega_+^T c - \Gamma_+)^T\end{aligned}$$

where $\Omega_+ = \Omega(T+1)$, $\Gamma_+ = \Gamma(T+1)$

We can use the same idea with several affine constraints instead of one

caveat: the recursive approach is still suboptimal, since we make at the outset a big approximation

(we encapsulate all knwoledge about the past in one single ellipsoid)