Lecture 30
Set simulation

– purpose
– set simulation
– robust filtering
– finite-horizon robust control

Purpose

The purpose of **set simulation** is to find bounds on trajectories of a dynamical process, given

– Uncertainty on initial condition
– Uncertainty of exact system parameters

Stochastic equivalent: if a dynamical system has “noise”, then trajectories “diffuse” with time

In the **robust filtering** problem, we assume that a (possibly noisy) measurement is taken at each time step (should help us estimate x)

In the **robust control** problem, we assume that at each step we are able to modify some system parameters, to “drive” trajectories

Discrete-time uncertain system

assume system described by

$$x_{k+1} = \mathbf{A}_k x_k + \mathbf{b}_k, \quad k = 0, 1, 2, \ldots$$

where

- x_k is the state (a n-vector)
- matrix $[\mathbf{A}_k \; \mathbf{b}_k]$ is uncertain, *i.e.*

  $$[\mathbf{A}_k \; \mathbf{b}_k] \in \mathcal{U}_k,$$

  where $\mathcal{U}_k$ is a given (algebraic) set

Set simulation: ellipsoidal approach

Assume x_0 is known (this is done WLOG: prove it!)

Ellipsoidal approach: given

- uncertainty model (*i.e.*, set $\mathcal{U}_k$)
- current ellipsoid of confidence for x_k, $\mathcal{E}_k$

compute $\mathcal{E}_{k+1}$ that is *guaranteed* to contain x_{k+1}

For now, we drop index k, so x is current state, x_+ is new state, etc

Parametrization of ellipsoids

For convenience (see why later), choose to parametrize **current** ellipsoid with Cholesky factor

$$\mathcal{E} := \{\xi = \hat{x} + Ez \mid \|z\|_2 \le 1\},$$

and new ellipsoid with shape matrix

$$\mathcal{E}_+ := \left\{\xi \;\middle|\; P_+ \succeq (\xi - \hat{x}_+)(\xi - \hat{x}_+)^T\right\}$$

where $P_+ \succeq 0$ and $\hat{x}, \hat{x}_+$ are the centers

With this convention: state update equation reads

$$\begin{bmatrix} x_+ \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} \begin{bmatrix} E & \hat{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \|z\|_2 \le 1$$

so WLOG we set $E = I$, $\hat{x} = 0$, $z = x$

Watch out: new matrix $\mathbf{A}$ is no longer square! (x_+ and x don't have same dimensions)

Overview of solution

Conceptually, the problem is as follows:

minimize **Tr** P_+ subject to

$$\begin{bmatrix} P_+ & \mathbf{A}x + \mathbf{b} - \hat{x}_+ \\ (\mathbf{A}x + \mathbf{b} - \hat{x}_+)^T & 1 \end{bmatrix} \succeq 0$$
$$\text{for every } x \in \mathcal{E}, \quad [\mathbf{A}\ \mathbf{b}] \in \mathcal{U}$$

- amenable to robust SDP technique ($x \leftrightarrow$ uncertainty)
- results in a confidence ellipsoid (above cond. is *guaranteed*)
- exact results for simple (eg, additive) perturbation structures

Case with no uncertainty

Assume first $\mathbf{A} = A$, $\mathbf{b} = b$ are exactly known (*i.e.*, set $\mathcal{U}$ is a singleton)

Assume current state (denoted x) is only known up to an ellipsoid; new state (denoted x_+) given by

$$x_+ = Ax + b$$

We seek to find an outer ellipsoidal approximation to the set of reachable states, *i.e.*: find the "best" ellipsoid $\mathcal{E}_+$ such that

$$\{Ax + b \mid x \in \mathcal{E}\} \subseteq \mathcal{E}_+$$

Solution is trivial: when A, b have no uncertainty, reachable set is again an ellipsoid!

Ellipsoid update: no uncertainty

problem: we want

$$(x_+ - \hat{x}_+)^T P_+^{-1} (x_+ - \hat{x}_+) \leq 1$$

to hold whenever

$$x_+ = Ax + b, \quad \|x\|_2 \leq 1.$$

i.e., one quadratic constraint must hold whenever another holds:

$$(Ax + b - \hat{x}_+)^T P_+^{-1} (Ax + b - \hat{x}_+) \leq 1 \text{ whenever } x^T x \leq 1$$

Using Lagrange relaxation: above condition is true iff there exist μ such that

$$\left[\begin{array}{c|cc} P_+ & A & b - \hat{x}_+ \\ \hline A^T & \mu I & 0 \\ (b - \hat{x}_+)^T & 0 & 1 - \mu \end{array}\right] \succeq 0.$$

Case with uncertainty

Robust ellipsoid update condition is

if there exist μ such that

$$\left[\begin{array}{c|cc} P_+ & \mathbf{A} & \mathbf{b}-\hat{x}_+ \\ \hline \mathbf{A}^T & \mu I & 0 \\ (\mathbf{b}-\hat{x}_+)^T & 0 & 1-\mu \end{array}\right] \succeq 0 \text{ for every } [\mathbf{A}\ \mathbf{b}] \in \mathcal{U},$$

then $\mathcal{E}_+$ is an ellipsoid of confidence for the new state

We can formulate sufficient conditions for the above to hold, using robust optimization

These conditions lead to an SDP in variables P_+, $\hat{x}_+$, μ and additional Lagrange multiplier (matrix) variable S

Example

continuous-time uncertain system:

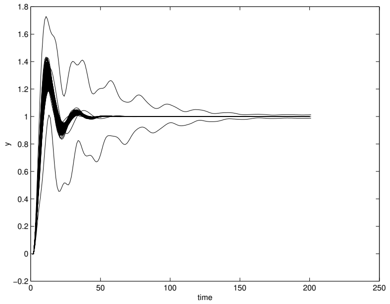
$$\ddot{y} + \mathbf{a}_1(t)\dot{y} + \mathbf{a}_2(t)y = \mathbf{a}_2(t),$$

where $\mathbf{a}_i(t) = a_i^{\mathrm{nom}}(1+\delta_i(t))$, with a_i^{nom}, $|\delta_i(t)| \leq \rho$, with $\rho \geq 0$ given

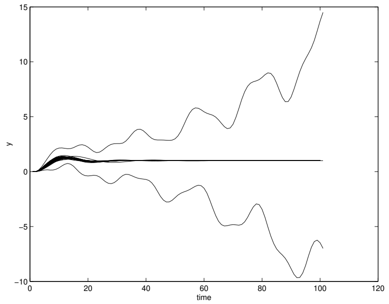
- discretize the system using forward-Euler
- form LFR of uncertain system

Set simulation

$\rho = 0.2$:

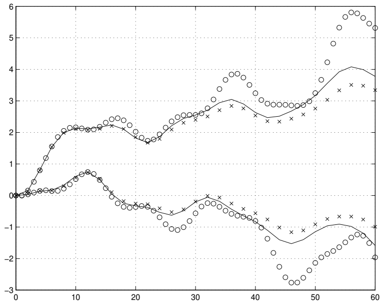


$\rho = 0.4$:



Constant uncertainty

If we know uncertainty is constant, we can use this information (homework)



Interval ARMA process

System described by an "interval AR" process

$$\xi(n+k) + \alpha_1(k)\xi(n+k-1) + \ldots + \alpha_n(k)\xi(k) = 0,$$

where time-varying α_i is only known to belong to a given interval $\alpha_0 + [-\rho\ \rho]$ (ρ measures absolute error level)

Choose order $n = 20$, x_0 given, and a specific value of α_0

Plot of output
$z(k) = c^T x = (\xi(k) + \ldots + \xi(n+k-1))/n$:

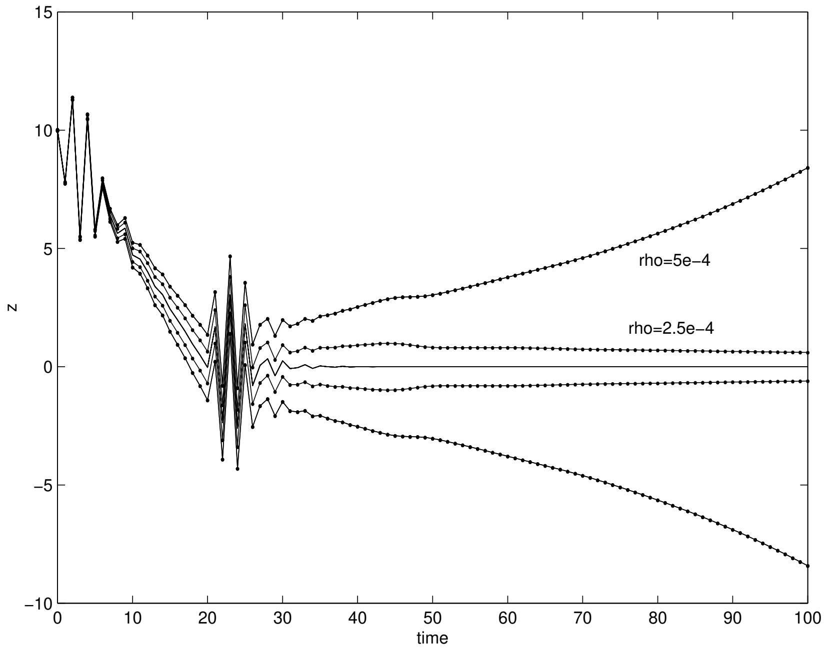
– obtain lower bound on value of z by computing

$$\min\ c^T x \quad x \in \mathcal{E}$$

where $c = [1 \ldots 1]/n$
answer: $z_{\min} = c^T \hat{x} - \|P^{1/2}c\|_2$

– similarly for upper bound

Set simulation



Robust collision detection

position of aircraft 1 relative to aircraft 2 described as

$$x_{k+1} = \mathbf{A}_k(\delta_k)x_k + \mathbf{b}_k(\delta_k), \quad k = 0, 1, 2, \ldots$$

where $[\mathbf{A}_k(\delta_k)\ \mathbf{b}_k(\delta_k)]$ is affine in δ_k, and $\|\delta_k\|_\infty \leq 1$.

plot relative trajectory (2D):