Lecture 31
Robust filtering and control

- Robust filtering
- Example
- Finite-horizon control

Robust filtering

Assume now that a measure y is available

$$\begin{aligned} x_{k+1} &= \mathbf{A}_k x_k + \mathbf{b}_k, , \quad k = 0, 1, 2, \ldots \\ y_k &= \mathbf{C}_k x_k + \mathbf{d}_k \end{aligned}$$

where

$$\begin{bmatrix} \mathbf{A}_k & \mathbf{b}_k \\ \mathbf{C}_k & \mathbf{d}_k \end{bmatrix} \in \mathcal{U}_k$$

is unknown-but-bounded

Filtering means here that at each time step we take some (noisy) measure y

This should help in the estimation of x, since a measure should reduce the size of the confidence ellipsoid

Change of notation

Again, we can assume that current ellispoid is a the unit sphere centered around zero, since if

$$\mathcal{E} = \{\xi = \hat{x} + Ez \mid \|z\|_2 \le 1\} ,$$

then we can rewrite state equation as

$$\begin{bmatrix} x_+ \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{C} & \mathbf{d} \end{bmatrix} \begin{bmatrix} E & \hat{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \|z\|_2 \le 1$$

From now, WLOG set $E = I$, $\hat{x} = 0$, and $z = x$

Robust filtering: principle

Idea is to go back to the case with no uncertainty, and derive an LMI condition for

$$(x_+ - \hat{x}_+)^T P_+^{-1} (x_+ - \hat{x}_+) \le 1$$

to hold whenever

$$x_+ = Ax + b, \quad 0 = Cx + d, \quad \|x\|_2 \le 1.$$

Turns out that in this case, Lagrange relaxation provides (again) exact results

Why? above equations define a (flat) ellipsoid, so we again have to enforce a quadratic constraint whenever another holds

Robust filtering: main result

$\mathcal{E}(P_{k+1}, \hat{x}_{k+1})$ is a new confidence ellipsoid iff $\exists \tau \geq 0$ such that

$$\begin{bmatrix} P_{k+1} & \mathbf{A}_k x + \mathbf{b}_k - \hat{x}_{k+1} \\ (\mathbf{A}_k x + \mathbf{b}_k - \hat{x}_{k+1})^T & 1 - \tau \|\mathbf{C}x + \mathbf{d} - y_k\|^2 \end{bmatrix} \succeq 0$$

$$\text{for every } x \in \mathcal{E}_k, \;\; [\mathbf{A}_k \; \mathbf{b}_k] \in \mathcal{U}_k$$

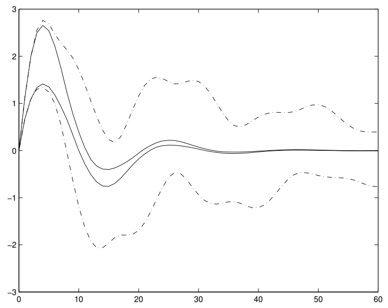
We can formulate sufficient conditions for the above to hold, using robust optimization

These conditions lead to an SDP in variables P_{k+1}, $\hat{x}_{k+1}$, μ and additional Lagrange multiplier (matrix) variable S

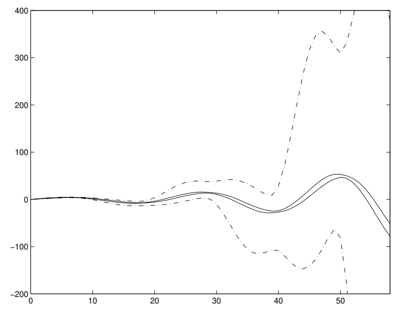
Example

second-order continuous-time uncertain system (page 616)

Set simulation (dotted) and filtering (solid) for a nominally stable uncertain system: ($\rho = 0.2$)



case of an unstable system: ($\rho = 0.4$)



Robust finite-horizon control

A controlled system:

$$\begin{aligned} x_{k+1} &= \mathbf{A}_k x_k + \mathbf{B}_k u_k, , \quad k = 0, 1, 2, \ldots \\ y_k &= \mathbf{C}_k x_k + \mathbf{D}_k u_k \end{aligned}$$

where

$$\begin{bmatrix} \mathbf{A}_k & \mathbf{B}_k \\ \mathbf{C}_k & \mathbf{D}_k \end{bmatrix} \in \mathcal{U}_k$$

is unknown-but-bounded, and u_k is a control variable

Purpose of robust control: design u_k so that the set of possible states lie in some region of trajectory space

Ellipsoidal approach: design u so that confidence ellipsoid lies in some desired region

Design constraints

Usually, we have constraints on u_k (*e.g.*, power limitations)

We can impose a variety of bounds on the confidence ellipsoid

Example:

- try to have the **center** of the ellipsoid follow, as closely as possible, a desired "target" trajectory
  (ensures a good "mean behavior")
- impose bounds, or minimize, the size of the confidence ellipsoid
  (ensures a "low variance")

many variations are possible

Robust control: main result

$\mathcal{E}(P_{k+1}, \hat{x}_{k+1})$ is a new confidence ellipsoid iff $\exists \tau \geq 0$ such that

$$\begin{bmatrix} P_{k+1} & \mathbf{A}_k x + \mathbf{B}_k u_k - \hat{x}_{k+1} \\ (\mathbf{A}_k x + \mathbf{B}_k u_k - \hat{x}_{k+1})^T & 1 - \tau \|\mathbf{C}x + \mathbf{D}u_k - y_k\|^2 \end{bmatrix} \succeq 0$$

$$\text{for every } x \in \mathcal{E}_k, \quad \begin{bmatrix} \mathbf{A}_k & \mathbf{B}_k \\ \mathbf{C}_k & \mathbf{D}_k \end{bmatrix} \in \mathcal{U}_k$$

Important point: above conditions are **convex** in u_k!

We can formulate sufficient conditions for the above to hold, using robust optimization

These conditions lead to an SDP in variables u_k, P_{k+1}, $\hat{x}_{k+1}$, μ and additional Lagrange multiplier (matrix) variable S