

Lecture 15 — Nov 6, 2008

Lecturer: Anant Sahai and David Tse

Scribe: Pulkit Grover

15.1 The relay channel

The relay channel is a communication channel with a sender and a receiver aided in communication by a relay node. A memoryless relay channel is specified by probability distribution $p(Y, Y_r|X, X_r)$, where X is the symbol transmitted by the source, X_r is the symbol transmitted by the relay, Y_r is the symbol received by the relay, and Y is the symbol received by the destination.

We first derive the following upper bound on the rate for the relay channel.

Theorem 15.1. *The capacity of the relay channel is upper bounded by*

$$C \leq C_{\text{cutset}} = \sup_{p(X, X_r)} \min\{I(X; Y_r, y|X_r), I(X, X_r; Y)\} \quad (15.1)$$

Proof: Let the blocklength be n . From Fano's inequality,

$$nR \leq I(W; \mathbf{Y}^n) + r\epsilon_r$$

Now,

$$\begin{aligned} I(W; \mathbf{Y}^n) &= \sum_{i=1}^n I(W; Y_i | \mathbf{Y}^{i-1}) \\ &\leq \sum_{i=1}^n H(Y_i | \mathbf{Y}^{i-1}) - H(Y_i | W, \mathbf{Y}^{i-1}) \\ &\leq \sum_{i=1}^n H(Y_i | \mathbf{Y}^{i-1}) - H(Y_i | W, \mathbf{Y}^{i-1}, X_i, X_{r_i}) \\ &= \sum_{i=1}^n H(Y_i | \mathbf{Y}^{i-1}) - H(Y_i | X_i, X_{r_i}) \\ &= \sum_{i=1}^n I(X_i, X_{r_i}; Y_i) \end{aligned}$$

This proves that the first of the two terms in the theorem forms an upper bound. To show the same for the second term,

$$\begin{aligned}
 I(W; \mathbf{Y}^n) &\leq I(W; \mathbf{Y}_r^n, \mathbf{Y}^n) \\
 &= \sum_{i=1}^n I(W; Y_{r_i}, Y_i | \mathbf{Y}_r^{i-1}, \mathbf{Y}^{i-1}) \\
 &= \sum_{i=1}^n I(W; Y_{r_i}, Y_i | \mathbf{Y}_r^{i-1}, \mathbf{Y}^{i-1}, x_{r_i}) \\
 &= \sum_{i=1}^n I(W; Y_{r_i}, Y_i | \mathbf{Y}_r^{i-1}, \mathbf{Y}^{i-1}, x_{r_i}) \\
 &\leq \sum_{i=1}^n H(Y_{r_i}, Y_i | x_{r_i}) - H(Y_{r_i}, Y_i | X_i, x_{r_i}) \\
 &= \sum_{i=1}^n H(Y_{r_i}, Y_i | x_{r_i}) - H(Y_{r_i}, Y_i | X_i, x_{r_i}) \\
 &= \sum_{i=1}^n I(X_i; Y_{r_i}, Y_i)
 \end{aligned}$$

The theorem follows using the usual trick of defining a time-sharing random variable Q uniform over $i = 1, \dots, r$. \square

15.2 A general relay network

In a general relay network, a transmitter and a receiver can be aided by multiple relays. For this general network, we the following theorem holds.

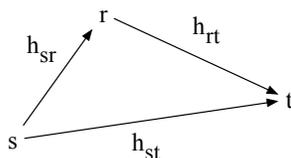


Figure 15.1. A noisy Gaussian single-relay channel

Theorem 15.2. For a general relay network with a single source and a single destination,

$$C < C_{\text{cutset}} = \sup_{p(x_1, \dots, x_n)} \min_{S: s \in S, t \in S^c} I(\mathbf{X}_s; \mathbf{Y}_{S^c} | \mathbf{X}_{S^c}) \quad (15.2)$$

Proof: Exercise! □

We first want to understand a noisy Gaussian single-relay network is shown in Fig. 15.1. The channel is given by

$$Y_r = h_{sr}X + Z_r$$

$$Y = h_{st}X + h_{rt}X_r + Z,$$

where Z_r and Z are distributed $\mathcal{N}(0, N_i)$. Without loss of generality, we assume that the noise power N_i and the transmit power P_i is 1 for all the nodes. The fade coefficients h_{sr} , h_{st} and h_{rt} can be suitably scaled if to get these powers to 1.

15.2.1 Linear deterministic network

Corollary 15.3. *For deterministic networks, the theorem evaluates to*

$$C = \min_{S: s \in S, t \in S^c} \text{rank}(G_{S \rightarrow S^c}) \log_2 |F|. \quad (15.3)$$

Proof: The upper bound is the cut-set bound, that is achievable using uniform distribution on the input alphabet. □

15.2.2 Deterministic relay networks

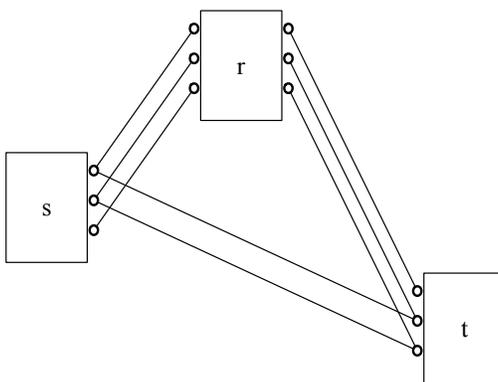


Figure 15.2. An example deterministic relay networks

Since we know the capacity region for deterministic relay channel, to gain understanding into good strategies for the Gaussian relay channel, we first consider the corresponding deterministic model instead.

Consider the deterministic model shown in Fig. 15.2. The cut-set bound for this network is

$$\begin{aligned} C_{cutset} &= \min\{\max(n_{st}, n_{sr}), \max(n_{st}, n_{rt})\} \\ &= n_{st} + \min\{(n_{sr} - n_{st})^+, (n_{rt} - n_{st})^+\} \end{aligned} \quad (15.4)$$

We will now show that this cutset bound is indeed achievable. Consider two cases

Case 1 : $n_{st} > \min\{n_{sr}, n_{rt}\}$.

Then turning off the relay achieves the first term in the cutset bound.

Case 2 : $n_{st} \leq \min\{n_{sr}, n_{rt}\}$ Decode and forward achieves the second term.

Can we generalize to Gaussian channel?

15.2.3 Generalization of decode and forward for general (noisy) single relay channel

The scheme is called block-Markov encoding. Intuition : Divide the time into subblocks of length n .

First block

- Source encodes a message W_1 , and transmits it.
- Relay decodes W_1 .
- The destination listens, but does not decode.

Second block

- The relay re-encodes W_1 and transmits it to destination t .
- The source encodes W_2 and transmits.
- Destination listens to relay, treating the source's signal as interference.
- Combine all these observations, and decode.
- Strip off W_1 's signal.

The scheme for the second block can be repeated as shown in Fig. 15.3.

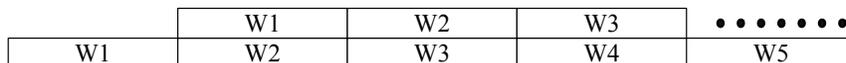


Figure 15.3. A block-Markov encoding scheme for Gaussian relay channel.

Performance analysis

A rate R achievable if the following two conditions are satisfied

1. W_1 can be decoded at the relay as long as

$$R < I(X; Y_r | X_r) \quad (15.5)$$

2. Has to be decodable by the final destination. This requires

$$R < I(W; Y | X_r) + I(X_r; Y) = I(X, X_r; Y), \quad (15.6)$$

This is obtained by first treating the source signal as interference, and decoding the relay signal in the current block, and then decoding the message W from the previous block using the relay signal from the previous block (see Fig. 15.3).

The achievable rate is, thus

$$R_{dec-fwd} = \max_{p(x)p(X_r)} \min\{I(X; Y_r | X_r), I(X, X_r; Y)\} \quad (15.7)$$

The cut-set bound is

$$C_{cut-set} = \max_{p(x, X_r)} \min\{I(X; Y_r, Y | X_r), I(X, X_r; Y)\} \quad (15.8)$$

Next time we will evaluate the two expressions for the Gaussian channel.