EECS 290S: Network Information Flow	Fall 2008

Lecture 21 — Nov 6, 2008

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21.1 Gaussian relay channels

In the last lecture we saw that for relay channels, the achievable rate is

$$R_{dec-fwd} = \max_{p(x)p(X_r)} \min\{I(X;Y_r|X_r), I(X,X_r;Y)\}, \tag{21.1}$$

and the cut-set bound is

$$C_{cut-set} = \max_{p(x,X_r)} \min\{I(X;Y_r,Y|X_r), I(X,X_r;Y)\}. \tag{21.2}$$

A few remarks are in order. The difference between the upper bound and the lower bound can potentially be large due to two reasons. Looking at the two expressions, the second terms are the same (apart from the probability distributions that are being maximized over). The first term in the cut-set bound suggests that the decoder and the relay are allowed to cooperate arbitrarily ("beamforming"). This is not the case in the achievable strategy. The other, somewhat more interesting difference between the two bounds is that the cut-set bound allows for generating *arbitrary joint distributions*, while the achievable strategy generates codewords independently, and hence only allows for product distributions.

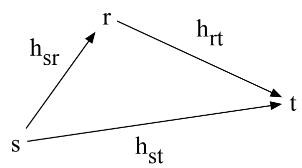

Figure 21.1. A noisy Gaussian single-relay channel

Lets evaluate these bounds for the Gaussian single-relay channel, shown in Fig. 21.1. We assume that the direct link is weak, otherwise we can simply ignore the relay. Thus,

$$\min\{|h_{sr}|, |h_{rt}|\} \geq |h_{st}|. \tag{21.3}$$

Gaussian distribution maximizes both of the terms in each of the lower and the upper bound. The lower bound turns out to be (we use complex Gaussians for consistency with the coming lectures)

$$R = \min\{\log(1+|h_{sr}|^2), \log(1+|h_{rt}|^2+|h_{st}|^2)\}. \tag{21.4}$$

The cut-set upper bound is given by

$$C_{cut-set} = \min\{\log(1 + |h_{sr}|^2 + |h_{st}|^2), \log(1 + (|h_{rt}| + |h_{st}|)^2)\}. \tag{21.5}$$

Now lets upper bound the cut-set bound. Using (21.3), the first term $\log(1+|h_{sr}|^2+|h_{st}|^2) \leq \log(1+2|h_{sr}|^2)$. Using the fact that for any two real numbers a and b, $(a+b)^2 \leq 2(a^2+b^2)$, the second term $\log(1+(|h_{rt}|+|h_{st}|)^2) \leq \log(1+2(|h_{rt}|^2+|h_{st}|^2))$. Thus,

$$C_{cut-set} \leq \min\{\log(1 + 2|h_{sr}|^2), \log(1 + 2(|h_{rt}|^2 + |h_{st}|^2))\}. \tag{21.6}$$

Comparing (21.4) and (21.6), the difference of the two bounds is at most one bit. To see this, lets take the difference of the first terms in both bounds

$$\begin{aligned}\log(1+2|h_{sr}|^2) - \log(1+|h_{sr}|^2) &= \log\left(\frac{1+2|h_{sr}|^2}{1+|h_{sr}|^2}\right) \\ &\leq \log\left(\frac{2+2|h_{sr}|^2}{1+|h_{sr}|^2}\right) \leq \log(2) = 1.\end{aligned}$$

Same proof works for the second term as well. Thus we have characterized the capacity of the Guassian single relay channel to within one bit. In fact, we know the the power required to achieve any rate R to within a factor of two. This characterization is stronger than the one bit characterization at low SNR.

A natural question is whether this the decode and forward scheme gets us within a constant number of bits for all single-source single-destination relay networks. The following example shows that this is not the case.

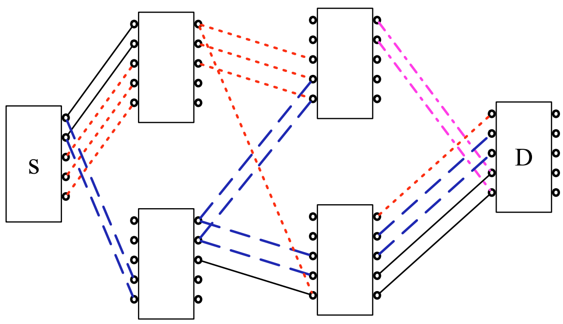

Figure 21.2. A deterministic network where decode and forward does not achieve the cut-set bound, but forwarding the received bits does.

For the network in Fig. 21.2, decode and forward does not achieve the cut-set bound. Instead, what is needed is to forward the received bits. This strategy is shown in the figure, where the flows are marked with their respective colors. The purple flow is the modulo-two sum of the blue flow and the red flow. Note that the top relay in the second layer cannot decode the blue bits nor the red bits, but it forwards the sum which is the purple flow.