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| **EECS 290S: Network Information Flow** | **Fall 2008** |

# Lecture 3 — September 4

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## Recap

- max flow = min cut. Proved last time.
- Showed a greedy algorithm that achieves this through path augmentation.
- Integer capacities lead to integer flows (readily generalize to rational numbers by rescaling capacities by LCD)

## Multicast Problem

**Goal: distribute the same info to all sources at max rate $R$.**

Pure routing multicast problem is Np hard.

Observation 1: $R$ is clearly upper bounded by

$$\min_i \text{mincut}(s, t_i)$$

We will show that this limit is always achievable through linear network coding.

## Linear Network Coding

### terminology

- Each symbol $x \in \mathbb{F}$ (Field $\mathbb{F}$ supports addition subtraction multiplication and nonzero division).
- Each link from node $i$ to node $j$ has capacity $c_{i,j}$, (can carry $c_{i,j}$ symbols per unit time.)
- Linear coding: we will associate a matrix, $\mathrm{A}_{i,j}$ with each edge $\in \mathbb{E}$.
- We denote the the input by a column vector

  $$\vec{x} = [x_1, \ldots, x_R]^T$$

  where $R$ is of the rate at which we want to communicate.

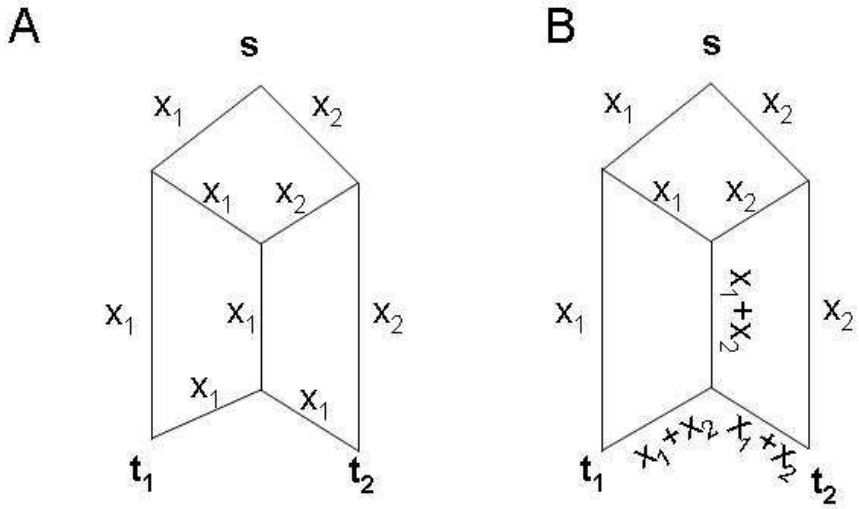


**Figure 3.1.** **A** Butterfly problem Routing Multicast solution or suboptimal information multicast. **B** Optimal information multicast.

## Structure

Each edge from the source is represented by a $c_{s,i} \times R$ matrix, $\mathrm{A}_{s,i}$. Evaluating $\mathrm{A}_{s,i}\vec{x}$ gives $\vec{x}_{s,i}$, the input to each of the $i$ nodes connected to the source. The output from a non-source node $i$ to node $j$, $\vec{x}_{i,j}$, is computed by evaluating,

$$\mathrm{A}_{i,j} \begin{bmatrix} \vec{x}_{k_1,i} \\ \vdots \\ \vec{x}_{k_\ell,i} \end{bmatrix}$$

where $x_{k_m,i}$ is the input to the non-source node $i$ from node $k_m$. The matrix $\mathrm{A}_{i,j}$ has therefor $c_{i,j}$ rows and $\sum_m c_{k_m,i}$ columns.

To illustrate this we compute the matrices associated with the butterfly network example in figure A.

For the rate $R$ to be achievable, we need to show the matrices at the receiving nodes have sufficient rank to be invertible and solve the linear system. Observe that the matrices for the example in figure B with $\alpha_1 = \alpha_2 = 1$ are,

$$t_1 : \mathrm{G}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$t_2 : \mathrm{G}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

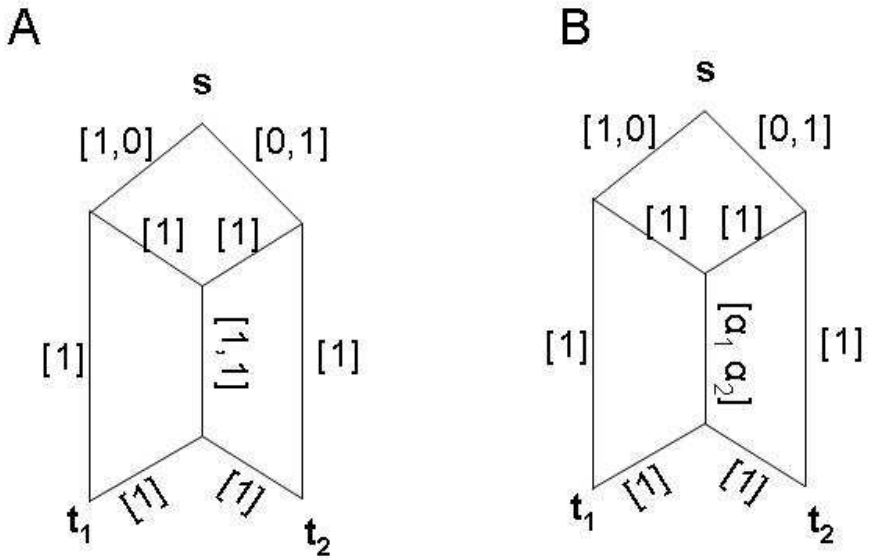


**Figure 3.2.** Edge matrices associated with simple butterfly network. **A** a solution. **B** general solution for $\alpha_1 \neq \alpha_2 \neq 0$

# achievability

We now assume acyclic networks for convenience. We will look at general networks later on.

**Fact:** Starting from the node $s$, you can order all nodes such that if $i$ is after $j$ in the ordering then there is no edge from $i$ to $j$

**Algorithm:** order the nodes by the maximum number of hops from $s$.

Given the local encoding matrices, we can compute the outputs at each node along this ordering sequentially, with the guarantee that all the inputs to a node are already computed when we get to that node. One can then construct end-to-end matrices $\mathrm{G}_1(\vec{\alpha}), \ldots \mathrm{G}_n(\vec{\alpha})$ for the destinations, where $\vec{\alpha}$ is the set of coefficients defining the local encoding matrices $\mathrm{A}_{i,j}$.

**Problem:** find $\{A_{i,j}\}$ such that $\mathrm{G}_1 \ldots \mathrm{G}_n$ are full rank. We will show this is achievable if

$$R \leq \min \operatorname{mincut}(s, t_i)$$

Note that under this condition, for all $i$, there exists $\vec{\alpha}^{(i)}$ such that $\mathrm{G}_i(\vec{\alpha}^{(i)})$ is invertible. (i.e. we can find a set of local encoding matrices that allows reconstruction of information for node $i$). This follows from our Ford-Fulkerson proof last time and the fact that the routing solution is a special case of a linear network code.

We need only to show that there is a choice of $\vec{\alpha}$ that simultaneously satisfies invertibility of all the end-to-end matrices. First we use the butterfly problem again to demonstrate intuitively that this intersection region is large.

Let us number the internal nodes from top to bottom and left to right sequentially 1-4. Consider the transmission matrix $[\alpha_1 \alpha_2]$ for the edge from node 3 to 4. Observe the resulting

matrices $G_1$ and $G_2$ are invertible for all choices of $\alpha_1, \alpha_2$ except the special cases where one of these terms is zero. Next time we shall prove more rigorously the 'good space' is large relative to the space of possible choices.