EECS 290S: Network Information Flow	Fall 2008

Lecture 9 — September 30

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9.1 Recap: Binary Expansion Model

See prior lecture notes.

9.1.1 Deterministic Relay Channel

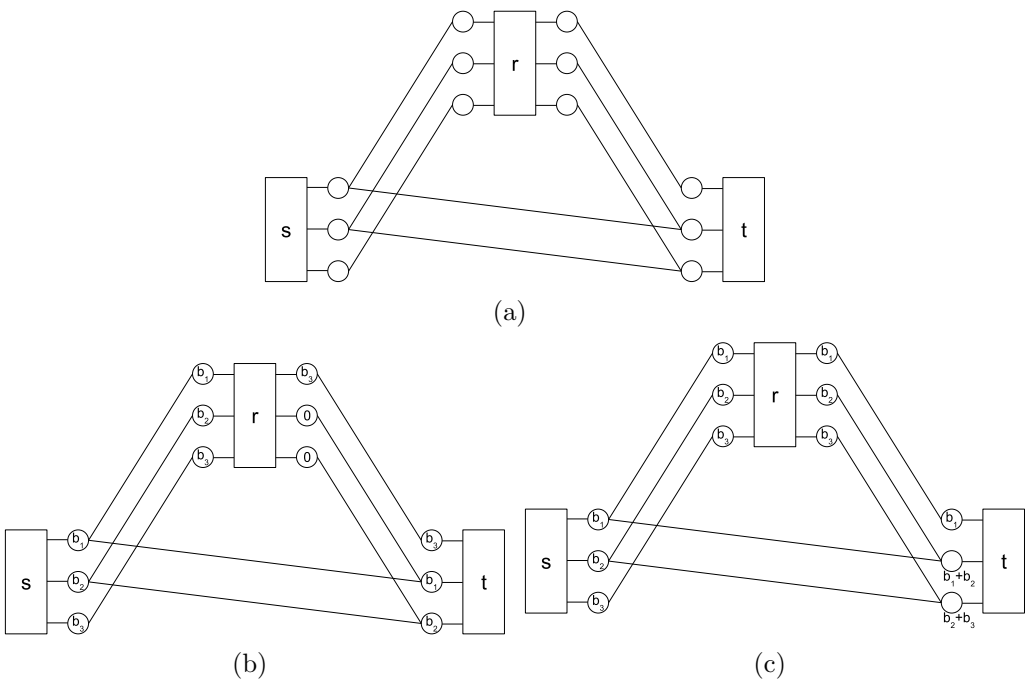

Figure 9.1. Channel Models: (a) Deterministic Relay Channel; (b) Achievable scheme 1: DF; (c) Achievable scheme 2: AF

In this example (Figure 9.1 (a)), cut-set upper bound: $C \leq 3$ bits/second. This bound is achievable by:

- Achievable scheme 1 (Figure 9.1 (b)) Decode and forward (DF): Relay decodes all 3 bits it receives and reorders the bits so that b_3 is transmitted at the highest level and nothing is sent at the lower two levels. Thus the destination node gets 3 clean information bits at 3 different levels.

- Achievable scheme 2 (Figure 9.1 (c)) Amplify and forward (AF): Relay simply forwards what it receives. Destination can figure out $b_2 = (b_1 + b_2) + b_1$ and $b_3 = (b_2 + b_3) + b_2$.

<u>**Exercise:**</u>

- **Show that if the relay does not talk and listen at the same time, one can achieve 2.5 bits/second.**
- **Generalize to arbitrary (n_{st}, n_{rt}, n_{sr}), and $\alpha =$ fraction of time relay listens.**
- **Solve for optimal α if the relay cannot listen and talk at the same time.**

9.1.2 Diamond Network

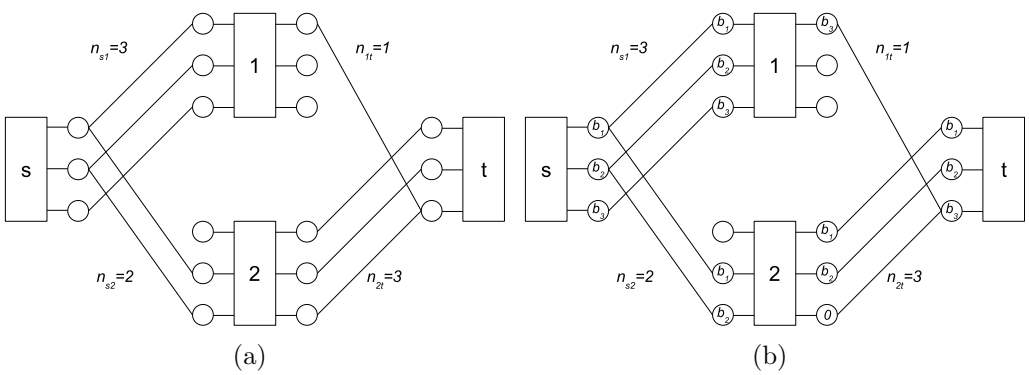

Figure 9.2. Channel Models: (a) Diamond Network; (b) Achievable scheme

In this example (Figure 9.2 (a)), cut-set upper bound:

$$C \leq \min\{\max(n_{s1}, n_{s2}), \max(n_{1t}, n_{2t}), n_{1t} + n_{s2}, n_{2t} + n_{s1}\} = \min\{3, 3, 3, 6\} = 3$$

Again, this bound is achievable by the achievable scheme described in Figure 9.2 (b). This time Relay 1 is doing DF, while Relay 2 is doing AF.

Received signal at node j:

$$\mathbf{y}_j = \sum_{(i,j)\in E} S^{q-n_{ij}} \mathbf{x}_i$$

If one sets $G_{ij} := S^{q-n_{ij}}$, one can generalize the above expression to

$$\mathbf{y}_j = \sum_{(i,j)\in E} G_{ij} \mathbf{x}_i$$

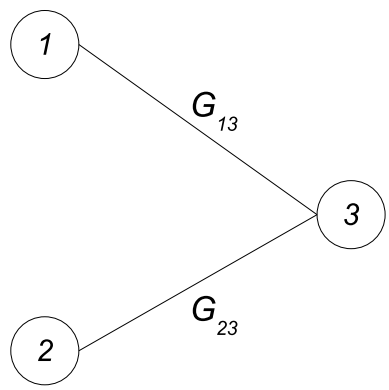

Figure 9.3. A Simple Wireline Network

where G_{ij} is an arbitrary matrix over the underlying field $F = \{0, 1\}$. For example, in the simple wireline network in Figure 9.3

$$\mathbf{y}_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = G_{13}\mathbf{x}_1 + G_{23}\mathbf{x}_2$$

9.2 Cut Value

We want to come up with a general cut-set upper bound for deterministic networks:

$$C \leq \min_S \operatorname{cut}(S, S^c)$$

Before that, we have to answer the question: what is the cut value $\operatorname{cut}(S, S^c)$?

- Wireline networks: $\operatorname{cut}(S, S^c)$ = sum of the capacities of the links crossing the cut.
- General linear: Define $G_{S \to S^c}$:= transfer matrix from left(S) to the right(S^c) of the cut.
  Conjecture: $\operatorname{cut}(S, S^c) := \operatorname{rank} G_{S \to S^c}$

<u>**Example**</u>: Diamond network (Figure 9.2 (a))

1) Take the cut: $S = \{s\}$, $S^c = \{1, 2, t\}$. Receive signal

$$\mathbf{y} = [b_1\ b_2\ b_3\ 0\ b_1\ b_2\ 0\ 0\ 0]^T = G_{S \to S^c}\,[b_1\ b_2\ b_3]^T\,,\ G_{S \to S^c} = \begin{bmatrix} G_{s1} \\ G_{s2} \\ G_{st} \end{bmatrix}$$

where $G_{s1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $G_{s2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $G_{st} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank $G_{S \to S^c} = 3$

2) Take the cut: $S = \{s, 1\}$, $S^c = \{2, t\}$. Receive signal

$$\mathbf{y} = [0\ b_1\ b_2\ 0\ 0\ c_1]^T = G_{S\to S^c}\,[b_1\ b_2\ b_3\ c_1\ c_2\ c_3]^T\ ,\ G_{S\to S^c} = \begin{bmatrix} G_{s2} & G_{12} \\ G_{st} & G_{1t} \end{bmatrix}$$

where $G_{s2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $G_{12} = G_{st} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $G_{1t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

rank $G_{S\to S^c} = 3$

Lemma 9.1 (Cut-set Upper Bound).

$$C \le \min_S \{\text{rank } G_{S\to S^c}\}$$

Proof: Refer to any basic network information theory text, eg., Cover and Thomas (second edition) Chapter 15.

However, by such techniques it is proved most easily by showing that the mutual information across such a cut is bounded by the rank above. This can be seen by just looking at the output entropy. The output entropy can be no more than the rank since a finite rank matrix can be written so that rank of the outputs depend on the inputs while the rest of the outputs depend only on those special outputs. The data-processing inequality then shows that it suffices to consider the mutual information from the input to the special bits, of which there are only rank. □

Theorem 9.2 (Capacity = Min-cut).

$$C = \min_S \{\text{rank } G_{S\to S^c}\}$$

This will be proved over the course of the next few lectures.