EE 290-T Lecture #3 Notes

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Topics:

1. Euclidean Geometry
2. Projective Geometry
3. Projective Space
4. 3D to 2D projection

1. Euclidean Geometry

- Euclidean geometry describes the world well.
- It allows to measure lengths and angles.
- Length, angles, parallelism, orthogonality, and all other properties that are related via a linear/Euclidean transform are preserved.
- Euclidean coordinates of a point in a plane are given by a 2-tuple ~ $[u, v]^T$.
- Ex: Consider the transformation that rotates 2 points, P_1, P_2, in a plane counter-clockwise θ° with respect to the origin as shown in Fig. 1. The transformation can be represented by the linear equations,

$$P_1' = R.P_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.P_1 \qquad 1.1$$

$$P_2' = R.P_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.P_2 \qquad 1.2$$

Since the transformation is Euclidean, the length between the two points, and the angle subtended at the origin, before and after the transformation remains the same.

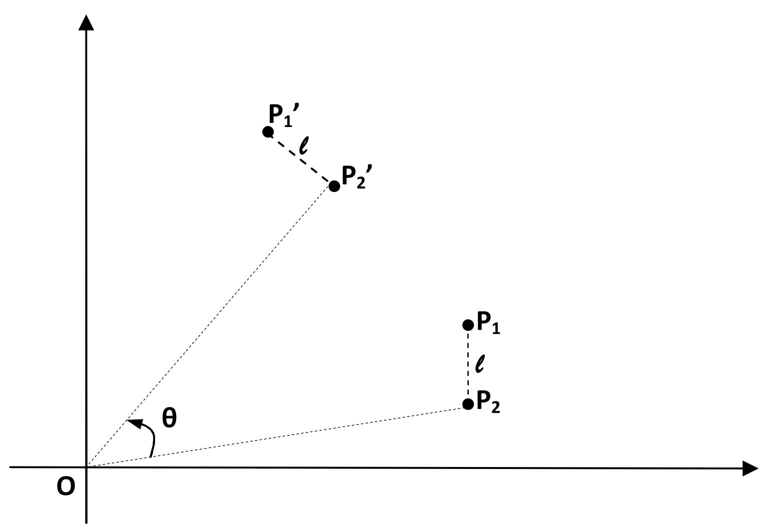

Fig. 1. – Rotating a point about the origin is a Euclidean transformation.

Q – Why do we need Projective geometry?

A – Because 3D objects are projected on to a 2D plane on capturing an image.

2. Projective Geometry

- Describes projection to lower dimensions well. For instance, parallel lines in 3D space are no longer parallel in a 2D image projection, and appear to meet. Such properties are captured well by projective geometry.
- The horizon has the same projection.
- Since parallelism between lines is not preserved, distances or angles are not preserved either.
- Projective geometry describes a larger class of transformations. It is an extension of Euclidean geometry and deals with the perspective projection of a camera.
- Projective coordinates of a point in a plane are homogenized and represented by a 3-tuple: $[u, v, 1]^T$.
- Rule: Scaling the projective coordinates by a non-zero factor does not change the Euclidean point it represents as it is homogenized. i.e., $[u, v, 1]^T \equiv [\lambda u, \lambda v, \lambda]^T$.

3. Projective Space

The Euclidean coordinates of a point in a plane can be represented by a 2-tuple: $[u, v]^T$. It's projective coordinates are obtained by appending a 1 to the vector: $[u, v, 1]^T$. By representing the point by this 3-tuple in projective coordinates, a one-to-one mapping is established between the 2D point in Euclidean coordinates and the corresponding point in projective coordinates. Thus, scaling the point by a non-zero zero factor does not change the Euclidean point it represents as it is homogenized. i.e., $[u, v, 1]^T \equiv [\lambda u, \lambda v, \lambda]^T$. Thus, projective coordinates represent naturally the operations performed by cameras.

Definition: The space of $(n+1)$-tuples of coordinates, with the rule that proportional (or scaled) $(n+1)$-tuples represent the same point, is called a *projective space* of dimension n, and is denoted $\mathbf{P}^n$.

In general, given coordinates in $\mathbf{R}^n$, the corresponding projective coordinates are obtained as,

$$[x_1, x_2, \dots, x_n]^T \xrightarrow{\mathbf{R}^n \to \mathbf{P}^n} [x_1, x_2, \dots, x_n, 1]^T. \qquad 3.1$$

To transform a point from projective coordinates back to Euclidean coordinates, we just need to divide by the last coordinate and the drop the last coordinate,

$$[x_1, x_2, \dots, x_n, x_{n+1}]^T \xrightarrow{\mathbf{P}^n \to \mathbf{R}^n} \left[\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right]^T. \qquad 3.2$$

Points with last coordinate $x_{n+1} \neq 0$ are usual points with representations in $\mathbf{R}^n$, but points of the form $[x_1, x_2, \dots, x_n, 0]^T$, do not have an equivalent representation in Euclidean coordinates. If we consider them as the limit of $[x_1, x_2, \dots, x_n, \lambda]^T$, when $\lambda \to 0$, (i.e. the limit of $[x_1/\lambda, x_2/\lambda, \dots, x_n/\lambda, 1]^T$) then they represent the limit of a point in $\mathbf{R}^n$ going to infinity in the direction $[x_1, x_2, \dots, x_n]^T$. Such points are called *points at infinity.*

Thus projective space contains more points than the Euclidean space of same dimensions, and is a union of the usual space $\mathbf{R}^n$ and the set of points at infinity. i.e.,

$$\mathbf{P}^n = \mathbf{R}^n \cup \{[x_1, x_2, \dots, x_n, 0]^T\}. \qquad 3.3$$

As a result of this formalism, points at infinity are represented without exceptions in projective space.

4. 3D to 2D Projection

Fig.2. represents the *pinhole model* of projection of a point in 3D onto the 2D *image plane, R*. Point ***C*** is the optical center, and does not belong to *R.* The projection, ***m***, of a point in 3D space, ***M***, is the intersection of the *optical ray*, (***C, M***), with the image plane. The *optical axis* is the line through ***C*** perpendicular to the image plane, pierces the image plane at the *principal point*, *c.*

The camera coordinate system is established by the orthonormal vectors, $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, centered at the optical center, ***C***. Here the image coordinate system is aligned with the camera coordinate system. The distance between ***C*** and *R* is the focal length, and is chosen as unity without loss of generality. Thus the relationship between the camera-coordinates of point ***M***=$[X, Y, Z]^T$, and the corresponding image-coordinates, ***m*** =$[u, v]^T$ is,

$$u = \frac{X}{Z}, \quad v = \frac{Y}{Z} \qquad 4.1$$

Once the projection has been captured by the image, the true 3D depth of the point ***M***, can no longer be inferred from a single image due to the inherent nature of 3D to 2D projection. Thus any other point, ***M'***=$[\lambda X, \lambda Y, \lambda Z]^T$, that lies on the optical ray (***C, M***), also has the same 2D-projection, ***m***. This depth ambiguity cannot be inferred from a single image of the point using geometry alone, and the only information available from the single image projection is the vector along which the 3D point lies in space.

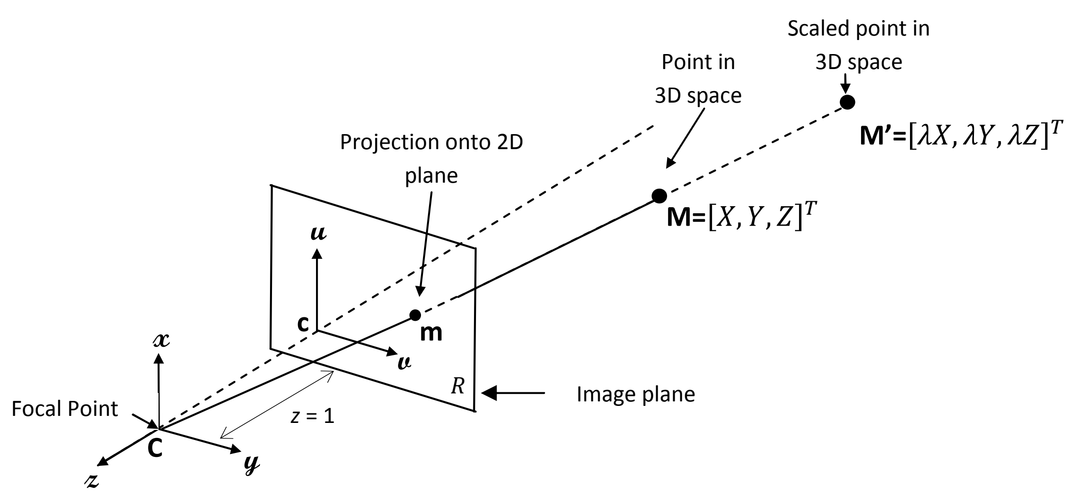

Fig.2. – Perspective projection of a 3D point onto a 2D image plane