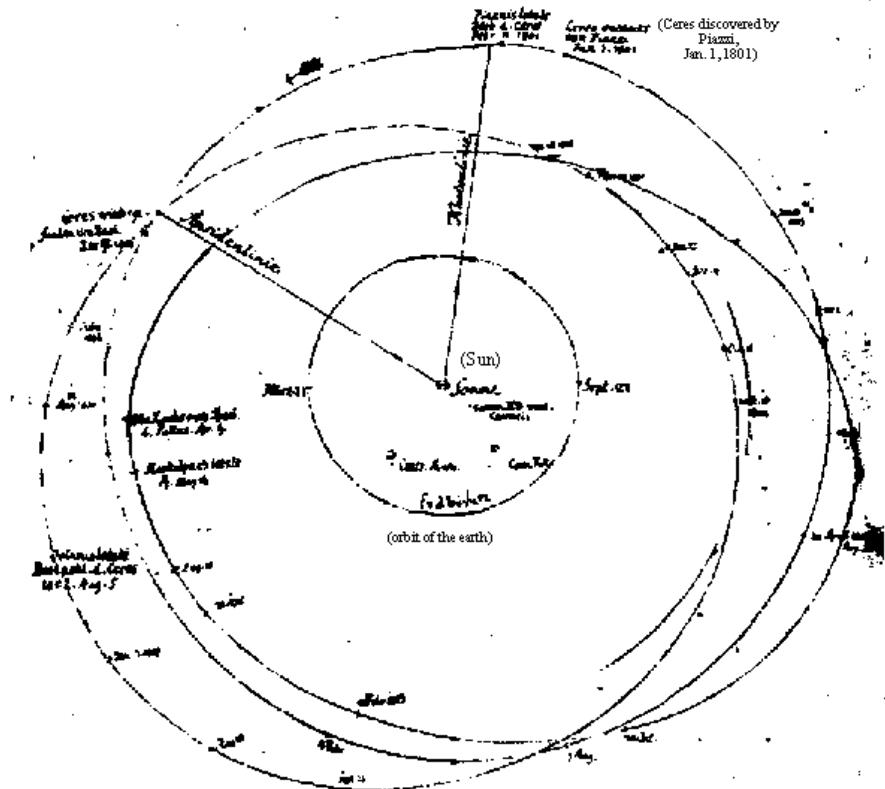


1809, Carl Friedrich Gauss



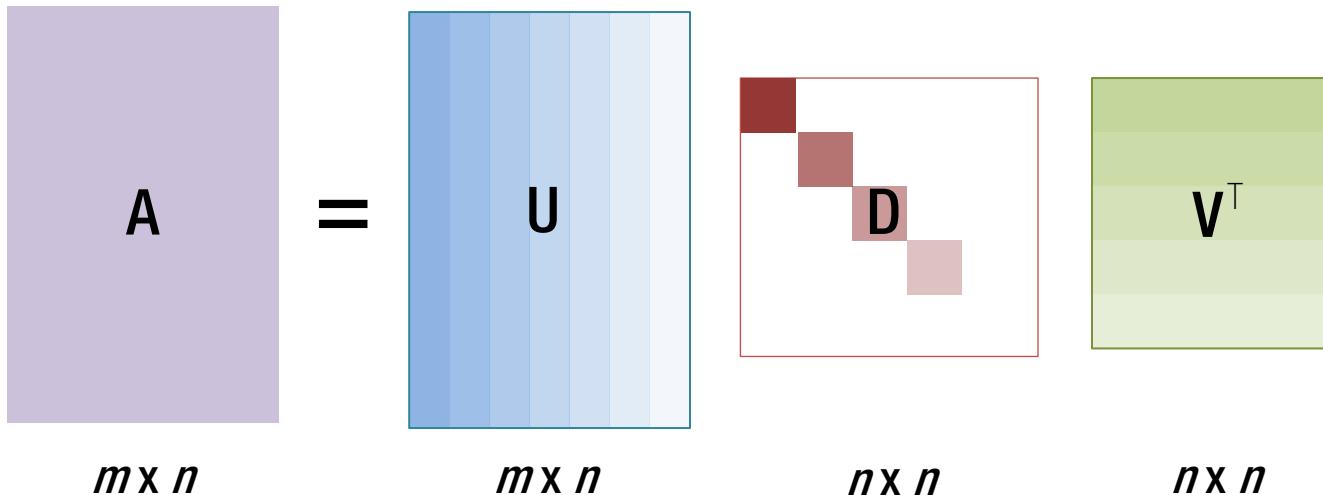
Sketch of the orbits of Ceres and Pallas (nachlaß Gauß, Handb. 4). Courtesy of Universitätsbibliothek Göttingen.

Singular Value Decomposition

$$A = U D V^T$$

The diagram illustrates the Singular Value Decomposition (SVD) of a matrix A . Matrix A is shown as a purple rectangle labeled A , with dimensions $m \times n$ below it. To its right is an equals sign ($=$). To the right of the equals sign are three matrices: U , D , and V^T . Matrix U is a blue rectangle with vertical stripes, labeled $m \times n$ below it. Matrix D is a red square matrix with a diagonal pattern of red squares, labeled $n \times n$ below it. Matrix V^T is a green rectangle with horizontal stripes, labeled $n \times n$ below it.

Singular Value Decomposition

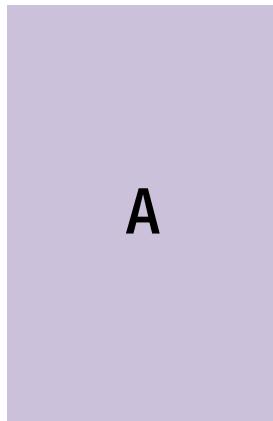


Column orthogonal matrix

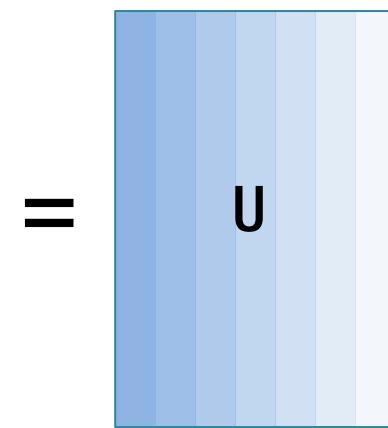
$$\mathbf{U}_i^\top \mathbf{U}_i = \|\mathbf{U}_i\| = 1$$

$$\mathbf{U}_j^\top \mathbf{U}_i = \mathbf{U}_j^\top \mathbf{U}_j = 0 \quad \text{for } i \neq j$$

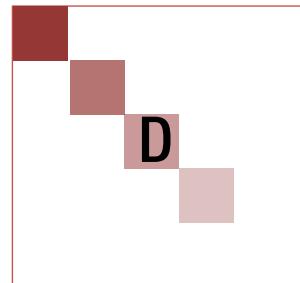
Singular Value Decomposition



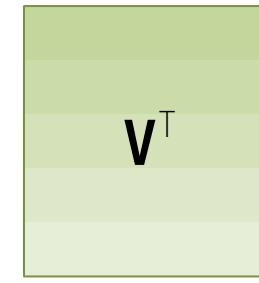
$m \times n$



$m \times n$



$n \times n$



$n \times n$

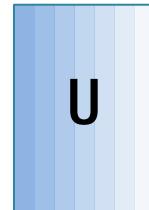
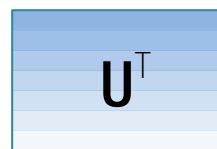
Column orthogonal matrix



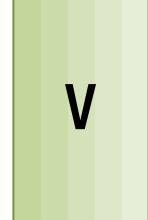
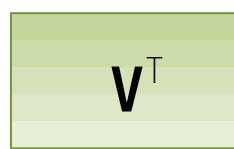
$$\|U_i\| = 1$$



$$U_i^T U_i = 0 \quad \text{for } i \neq j$$



$$U^T U = I_{m \times m}$$



$$V^T V = I_{n \times n}$$

Singular Value Decomposition

$$A = UDV^T$$

Diagram illustrating the Singular Value Decomposition (SVD) of a matrix A . The matrix A is shown as a purple rectangle labeled $m \times n$. It is equal to the product of three matrices: U (blue vertical rectangles), D (diagonal red squares), and V^T (green horizontal rectangles). All three matrices are labeled $m \times n$.

Singular value matrix

$$D = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\} \quad \text{where } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$$

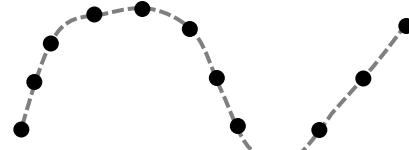
Diagram showing the singular value matrix D as a diagonal matrix with red squares. The matrix D is enclosed in a red border.

Singular Value Decomposition

$$A = U D V^T$$

Dimensions:

- $A: m \times n$
- $U: m \times n$
- $D: n \times n$
- $V^T: n \times n$



Basis

$$U_1$$

$$\beta_1 + \beta_2 + \beta_3 + \dots$$

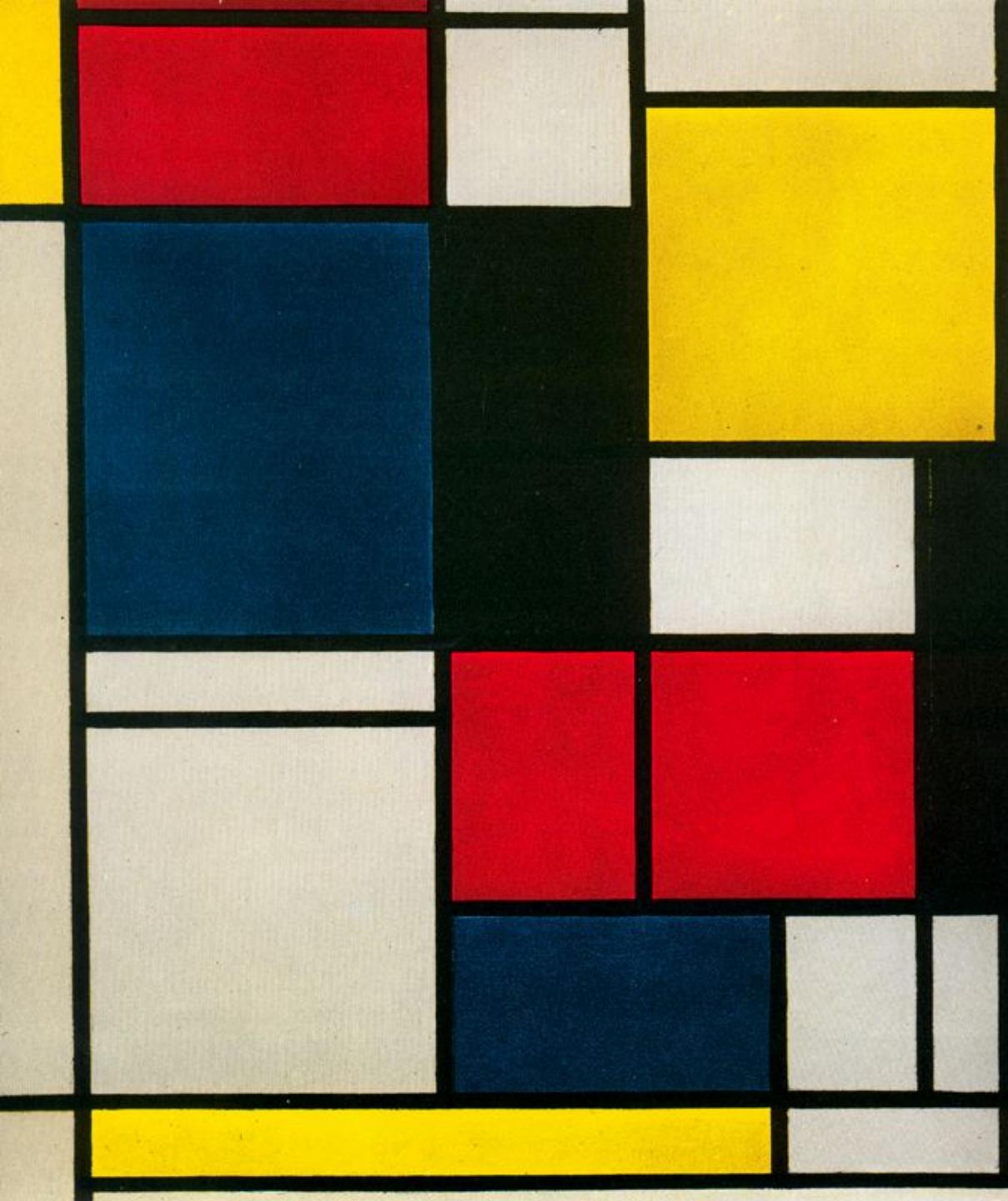
Scale

$$U_2$$

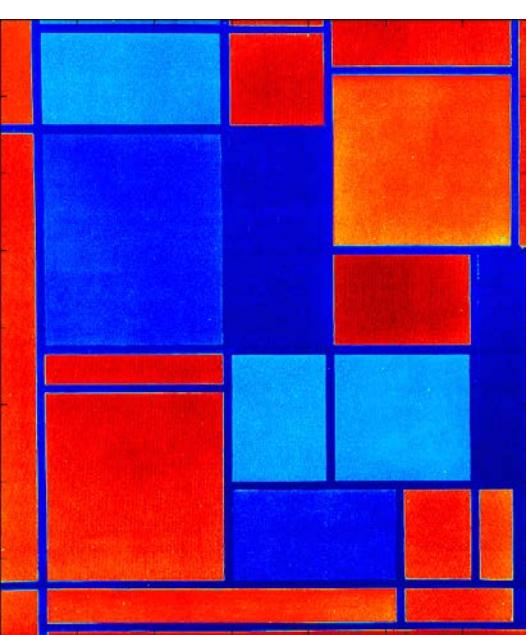
Address

$$U_3$$

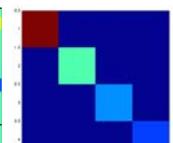
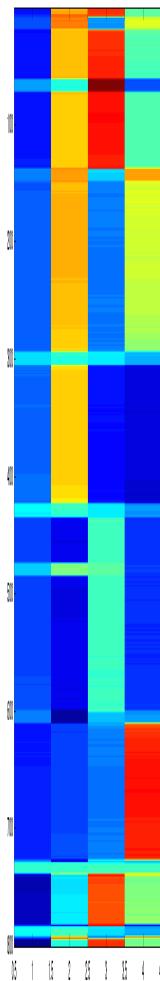
SVD as basis + transformed Address

A Piet Mondrian style abstract painting featuring a grid of black lines on a white background. Large rectangles in primary colors (red, blue, yellow) are placed within the grid. The composition includes a top row with a yellow rectangle on the right, a middle row with a blue rectangle on the left, and a bottom row with red and blue rectangles side-by-side.

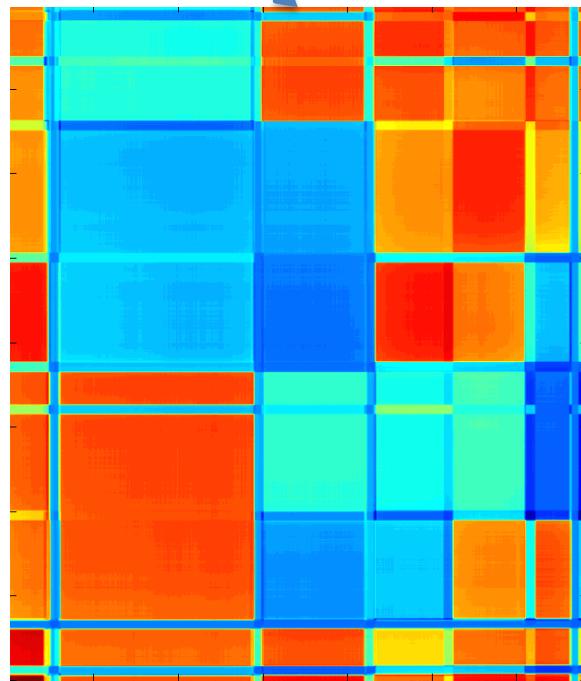
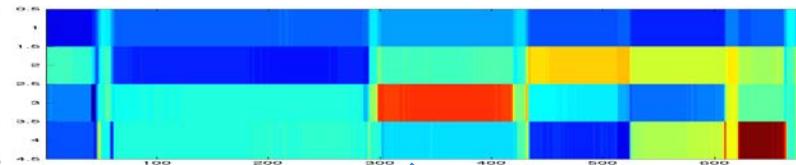
**SVD of
this?**

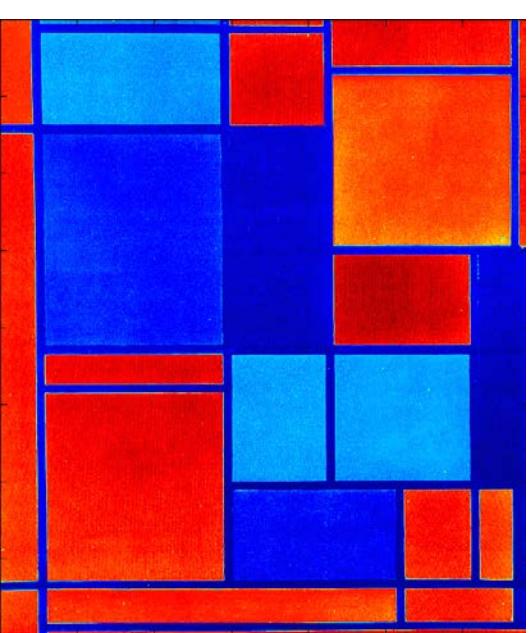


$U(:,1:4)$ $D(1:4,1:4)$

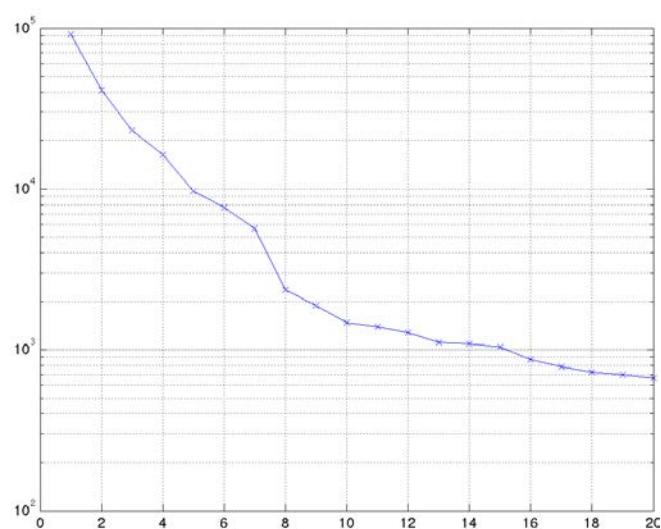


$V(:,1:4)'$



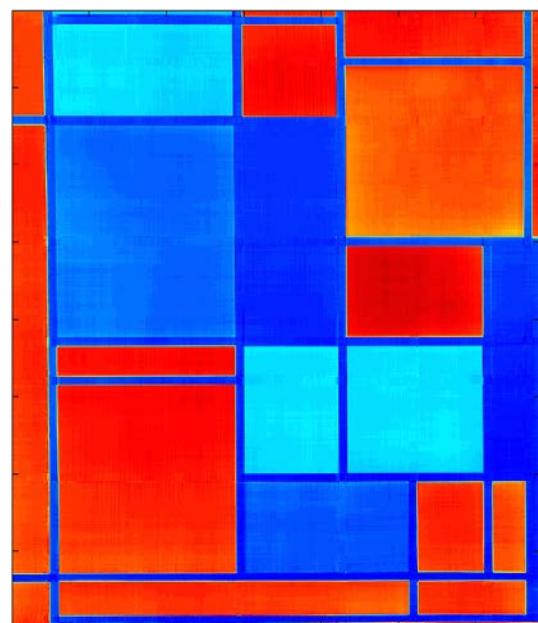


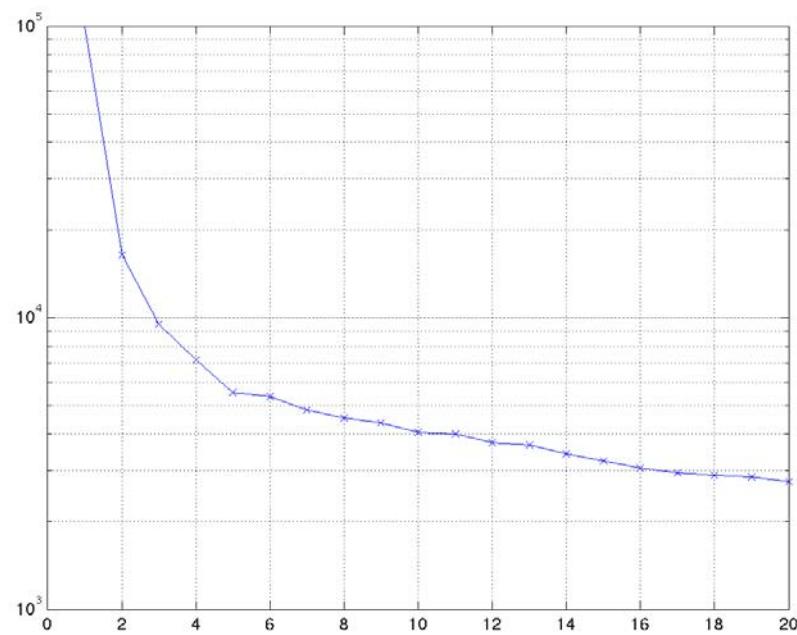
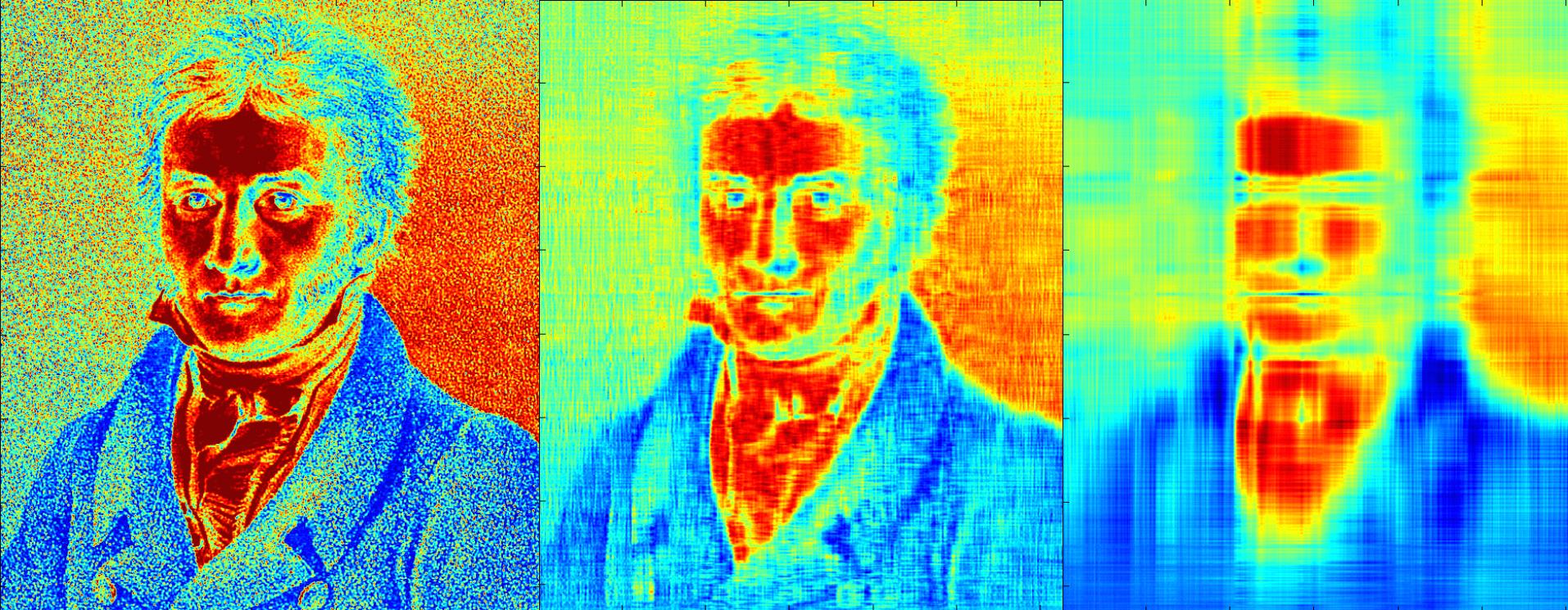
```
[u,d,v] = svd(l);
```

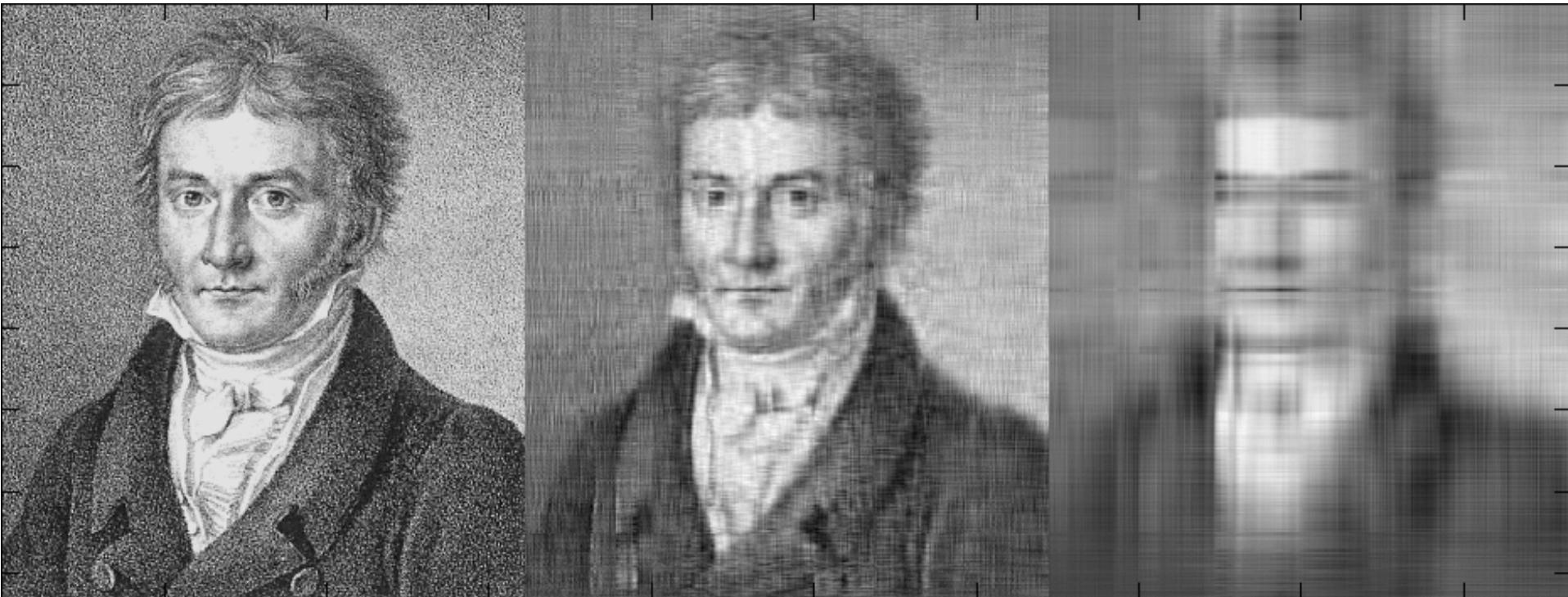


```
semilogy(diag(d(1:20,1:20)), 'x-')
```

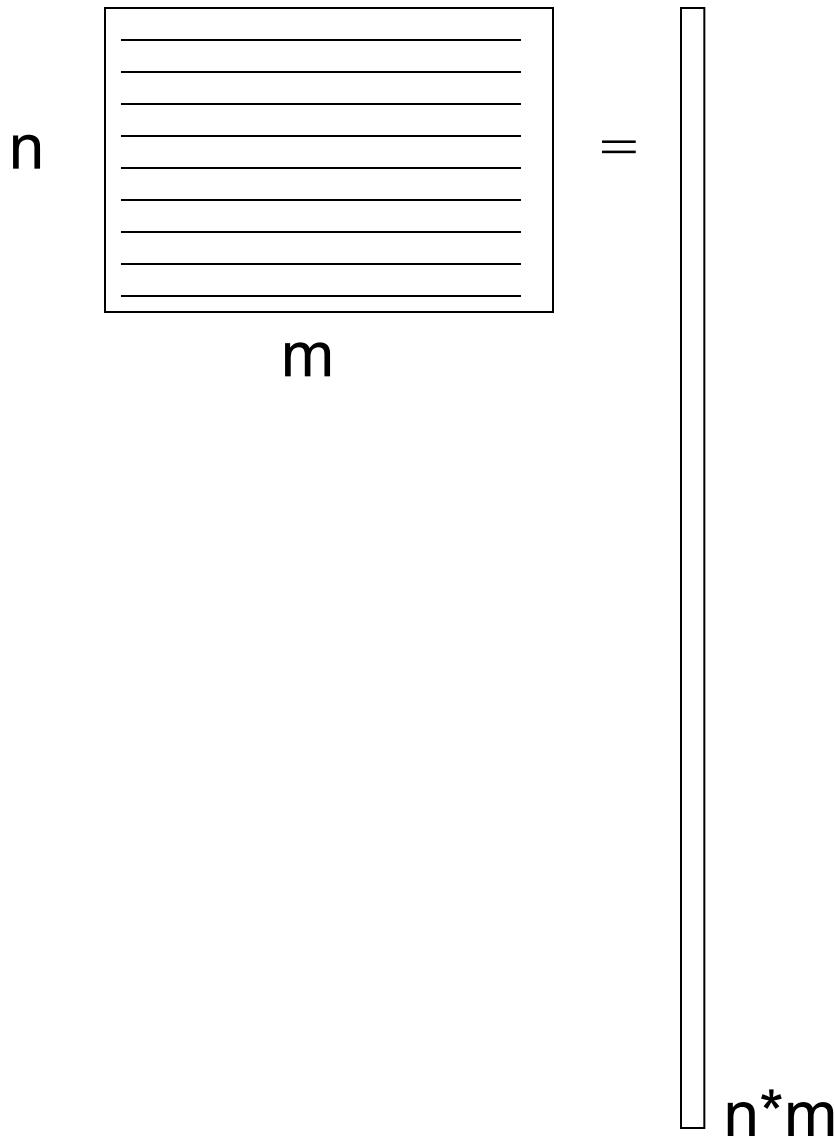
```
Im2 = u(:,1:20)*d(1:20,1:20)*v(:,1:20)';
```







Images as Vectors



Vector Mean

$$\begin{matrix} n \\ \text{---} \\ m \\ \text{---} \end{matrix} = \begin{matrix} | & | \\ \text{---} & \text{---} \\ | & | \end{matrix} + \begin{matrix} | & | \\ \text{---} & \text{---} \\ | & | \end{matrix} = \text{mean image}$$

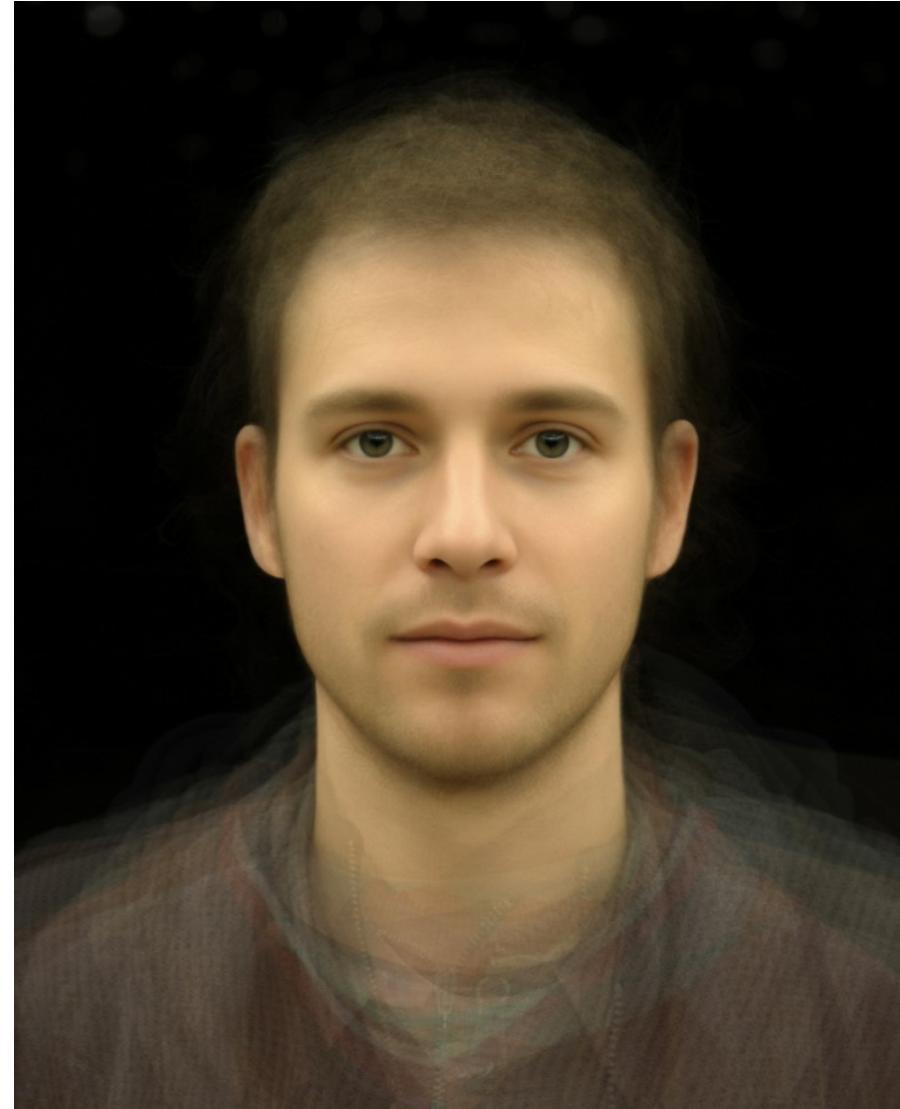
The diagram illustrates the calculation of a mean image from two input images, I_1 and I_2 .
Input I_1 is represented by a vertical rectangle of size $n \times m$, containing a grayscale house-like shape.
Input I_2 is represented by a vertical rectangle of size $n \times m$, containing a grayscale house-like shape.
The sum of I_1 and I_2 is shown as a vertical rectangle of size $n \times m$, resulting in a mean image where the house shape is darker than either I_1 or I_2 .

Eigenfaces



Eigenfaces look somewhat like generic faces.

Eigen-images of Berlin



Eigen-images



Average of 16 individuals transformed via
biometrical data of different ethnics



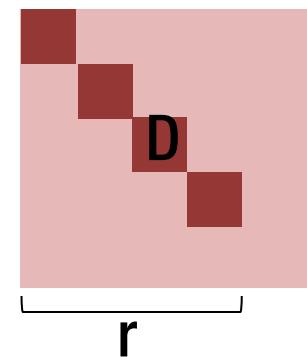
Average of 16 individuals transformed via
biometrical data of different ages

Rank

$$A = U D V^T$$

Diagram illustrating the Singular Value Decomposition (SVD) of a matrix A . The matrix A is shown as a purple $m \times n$ rectangle. It is equal to the product of three matrices: U (blue $m \times n$ rectangle), D (pink $n \times n$ diagonal matrix with red entries), and V^T (green $n \times n$ rectangle).

$$\text{rank}(A) = r \leq \min(m, n)$$



Rank is the same as the number of nonzero singular values

Nullspace

$$\begin{matrix} A & = & U & D \\ \textcolor{purple}{m \times n} & & \textcolor{blue}{m \times n} & \textcolor{red}{n \times n} \\ & & & \textcolor{green}{n \times n} \end{matrix}$$

$$\text{null}(A) = \begin{matrix} V_{r:n} \end{matrix} \longrightarrow \begin{matrix} A \\ \textcolor{purple}{m \times n} \end{matrix} \begin{matrix} V_{r:n} \end{matrix} = 0$$

Example III (Fundamental Matrix)

$F =$

```
1.0e+003 *
-0.0000  0.0000  0.0030
-0.0001  0.0002  0.0564
0.0132 -0.0292 -9.9998
```

$[u,d,v] = \text{svd}(F)$

$u =$

```
-0.0003  0.9981  0.0618
-0.0056 -0.0618  0.9981
1.0000 -0.0001  0.0056
```

$v =$

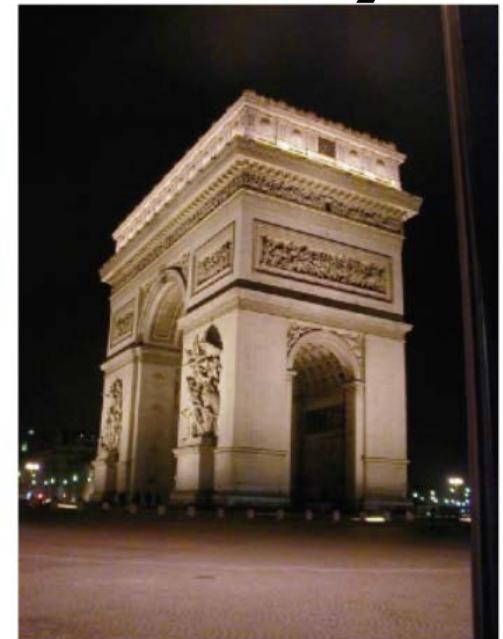
```
0.0013 -0.9660  0.2586
-0.0029 -0.2586 -0.9660
-1.0000 -0.0005  0.0032
```

$d(1,1)$

$\text{ans} = 1.0000e+004$

$d =$

```
1.0e+004 *
1.0000    0      0
0   0.0000    0
0     0    0.0000
```



$\text{Rank}(F) = 2$

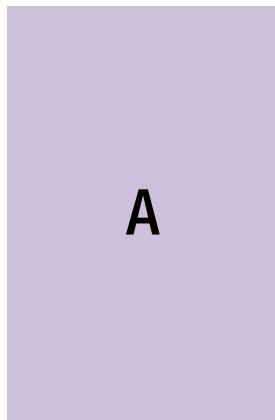
$d(2,2)$

$\text{ans} = 0.0021$

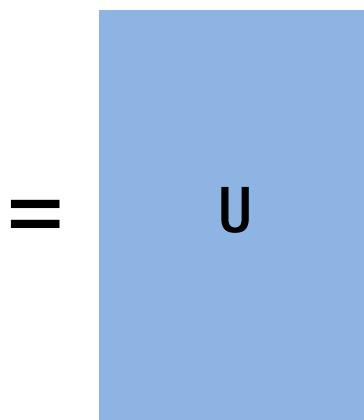
$d(3,3)$

$\text{ans} = 2.7838e-016$

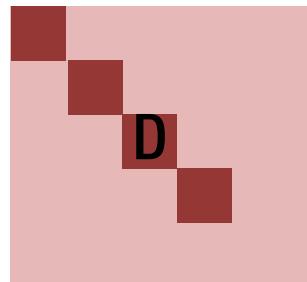
Matrix Inversion with SVD



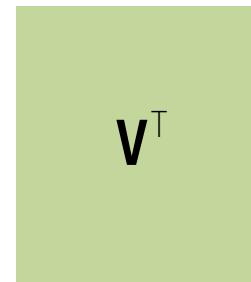
$m \times n$



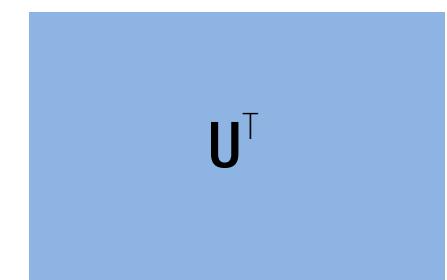
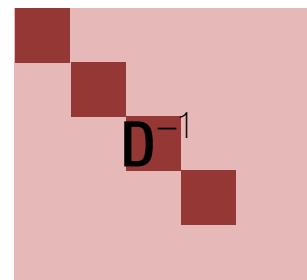
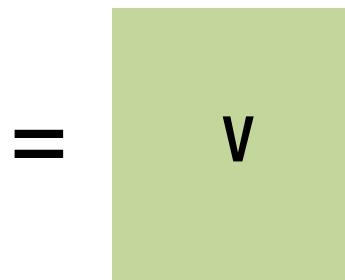
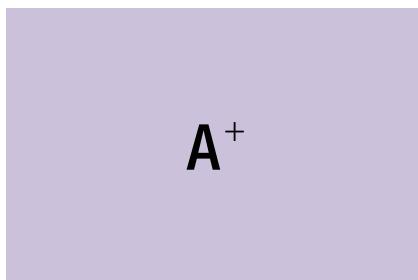
$m \times n$



$n \times n$



$n \times n$



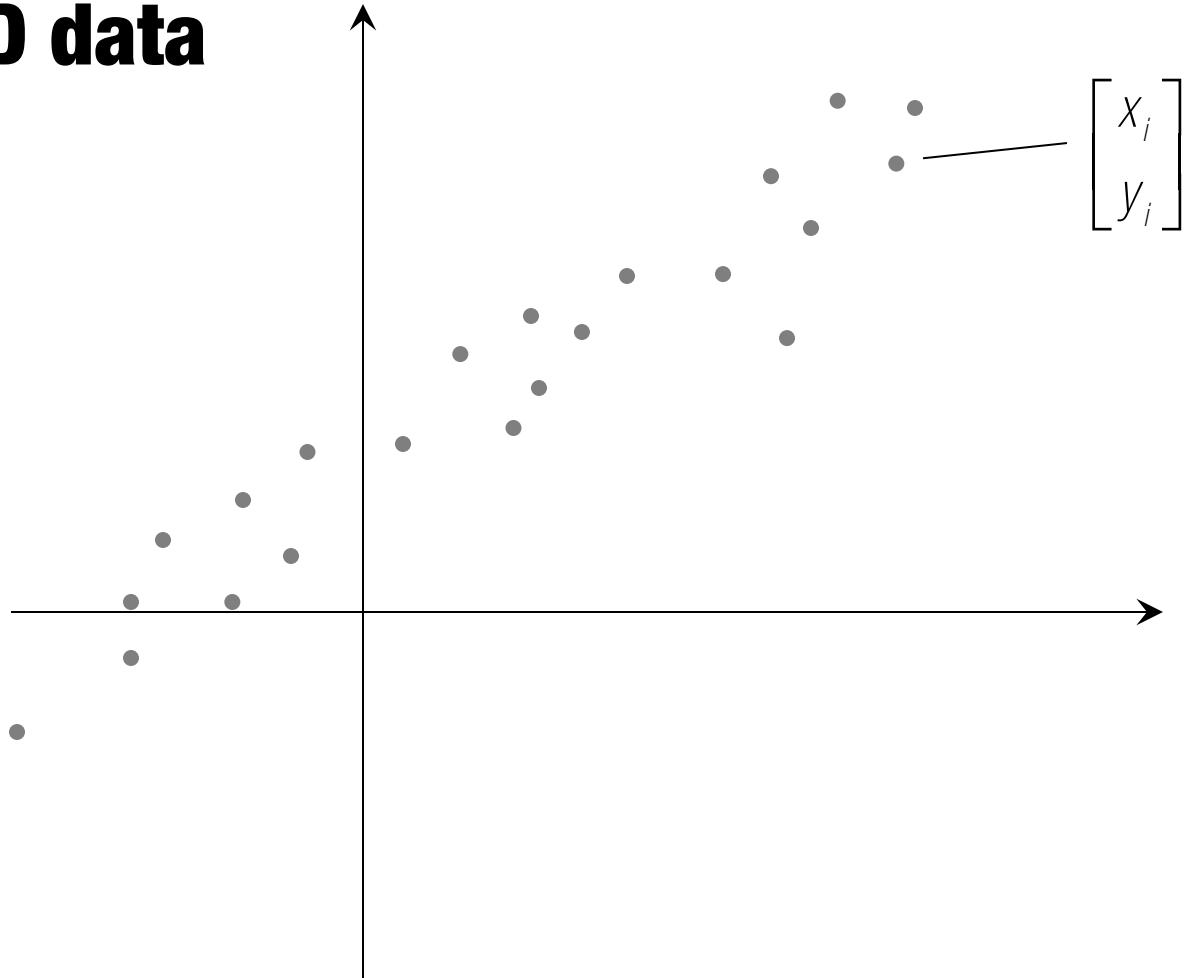
$$A^+ A = I \quad D^{-1} = \text{diag}\{1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_n\} \quad \text{if } \sigma_i > 0, \text{ otherwise zero.}$$

Two types of Least Square Problem:

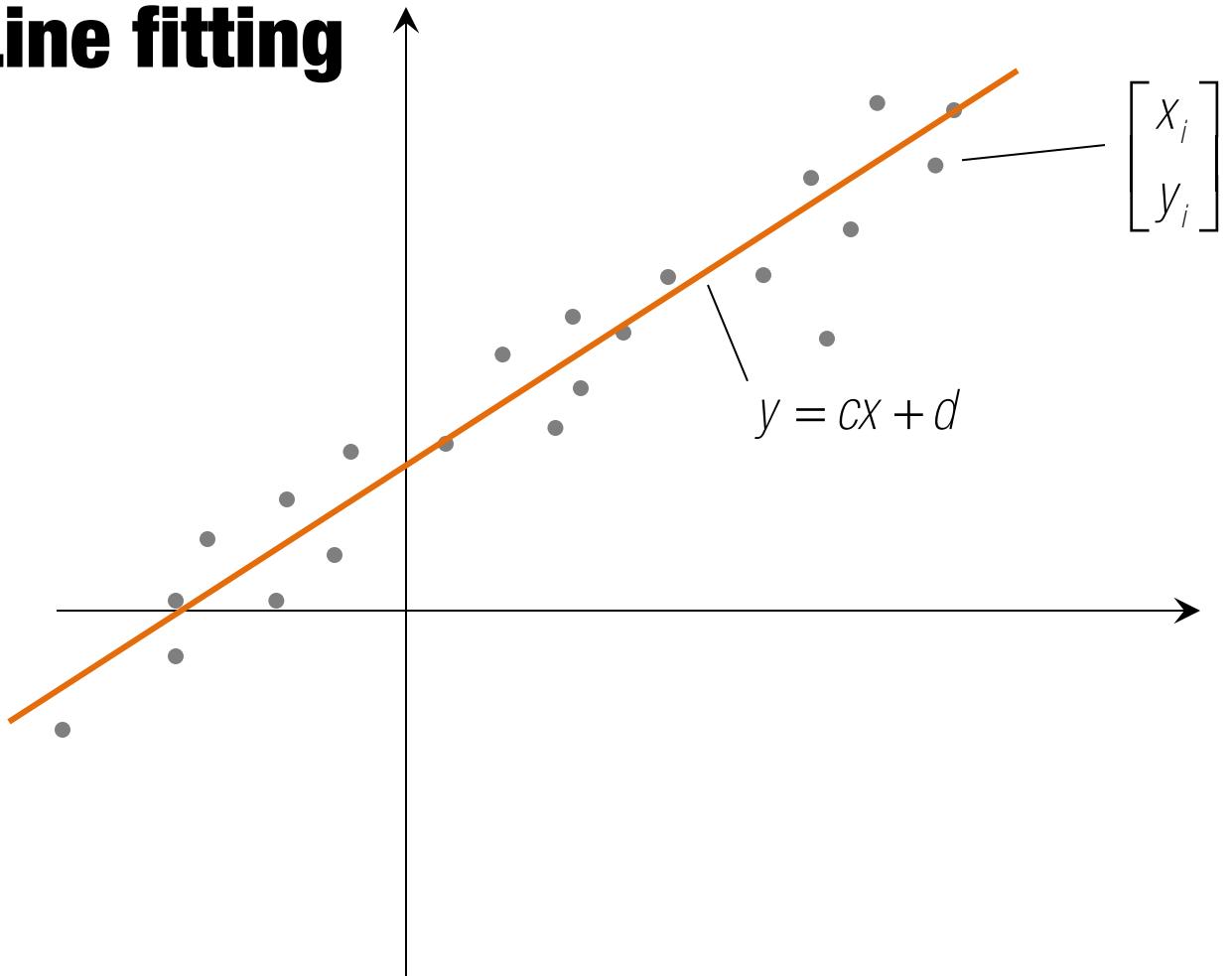
$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|^2 \quad \text{with} \quad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax}\|^2 \quad \text{s.t.} \quad \|\mathbf{x}\| = 1$$

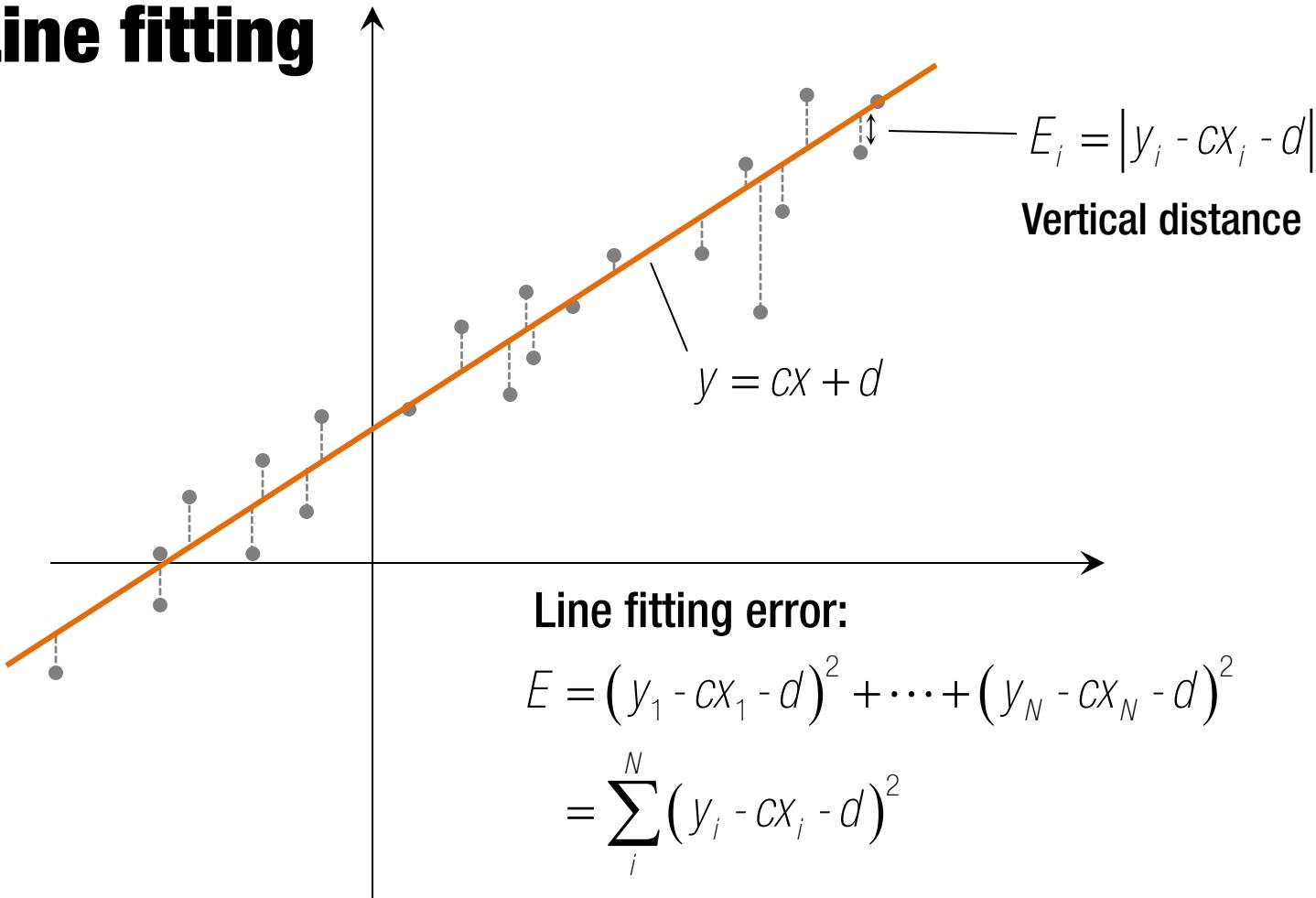
2D data



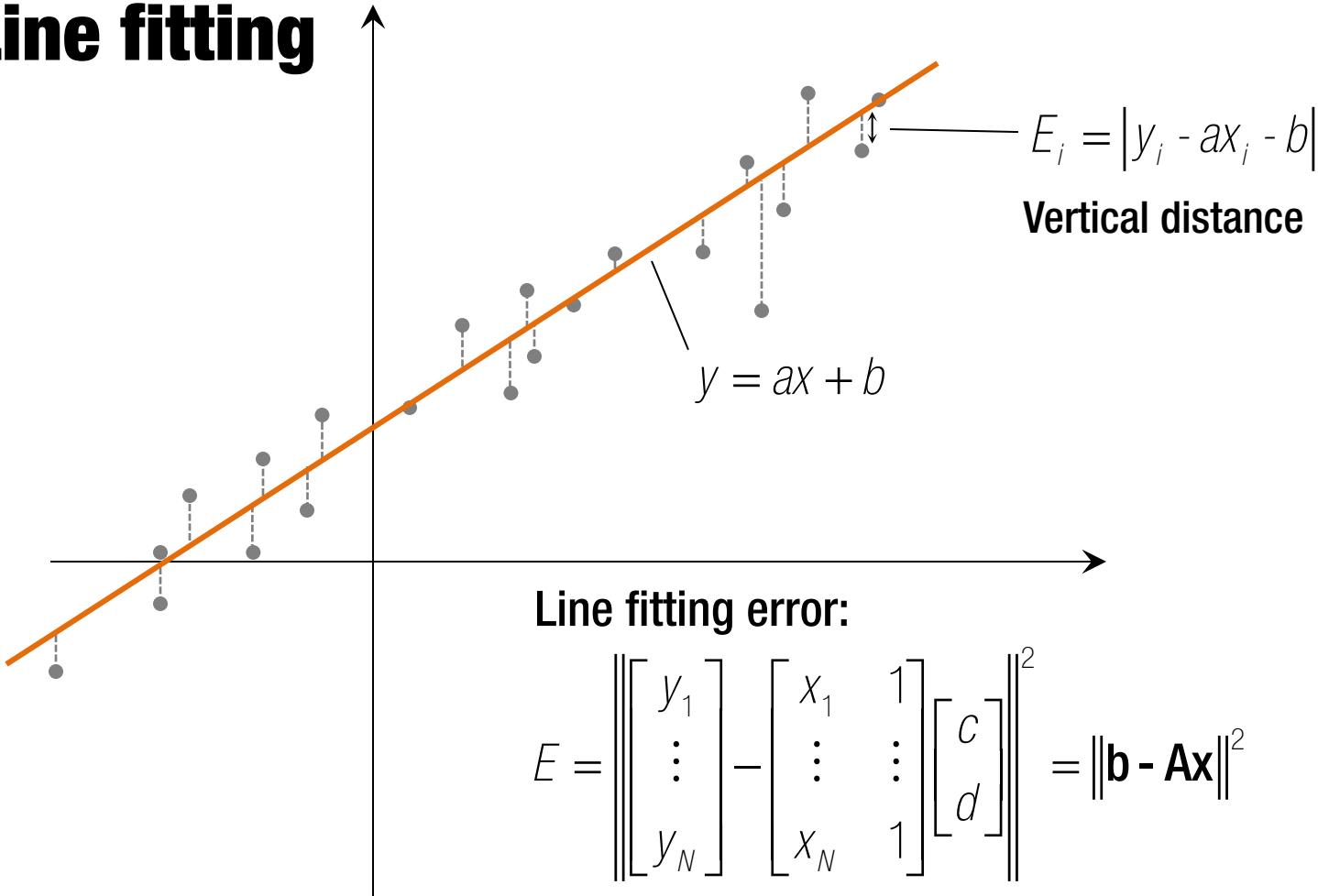
Line fitting



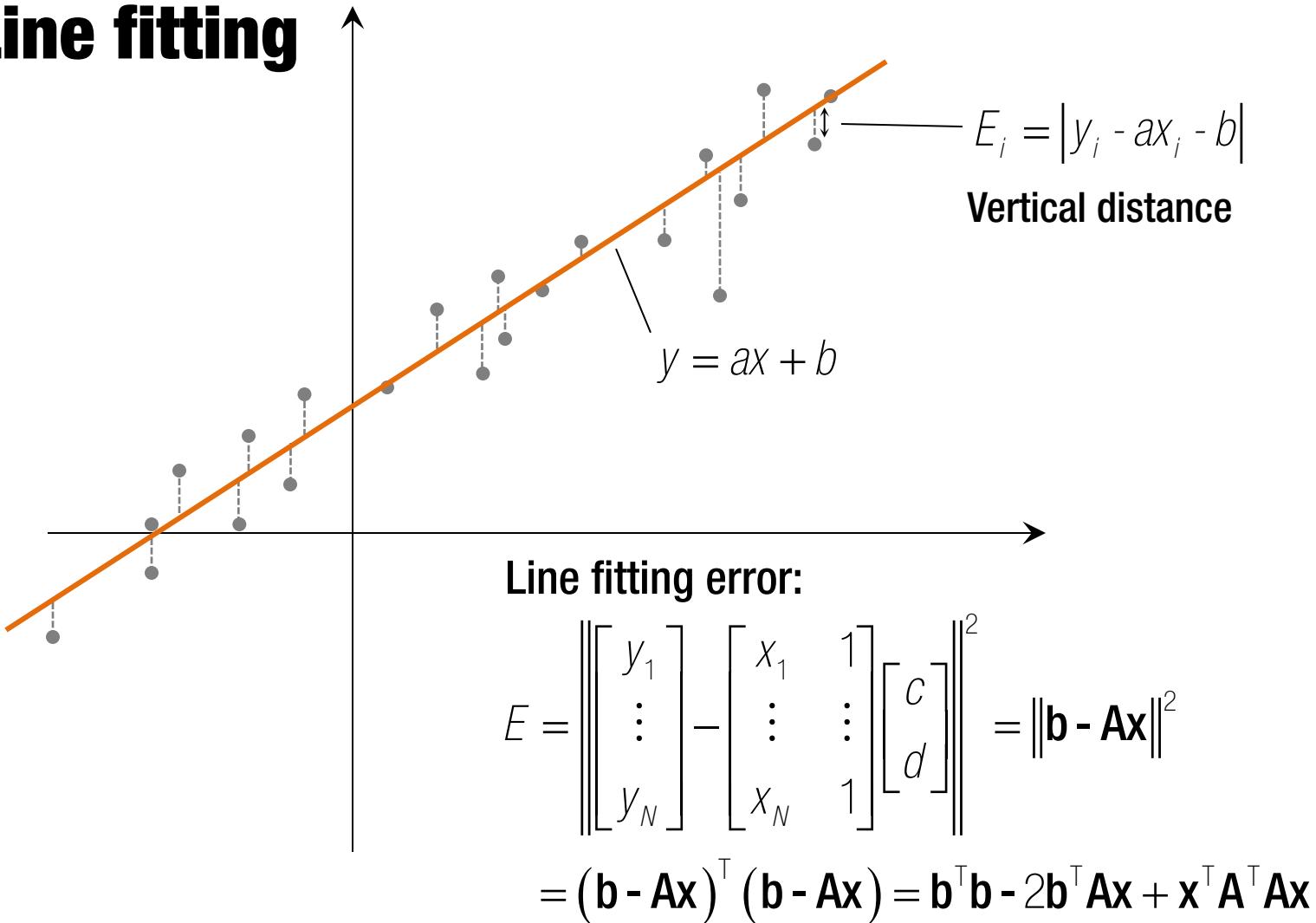
Line fitting



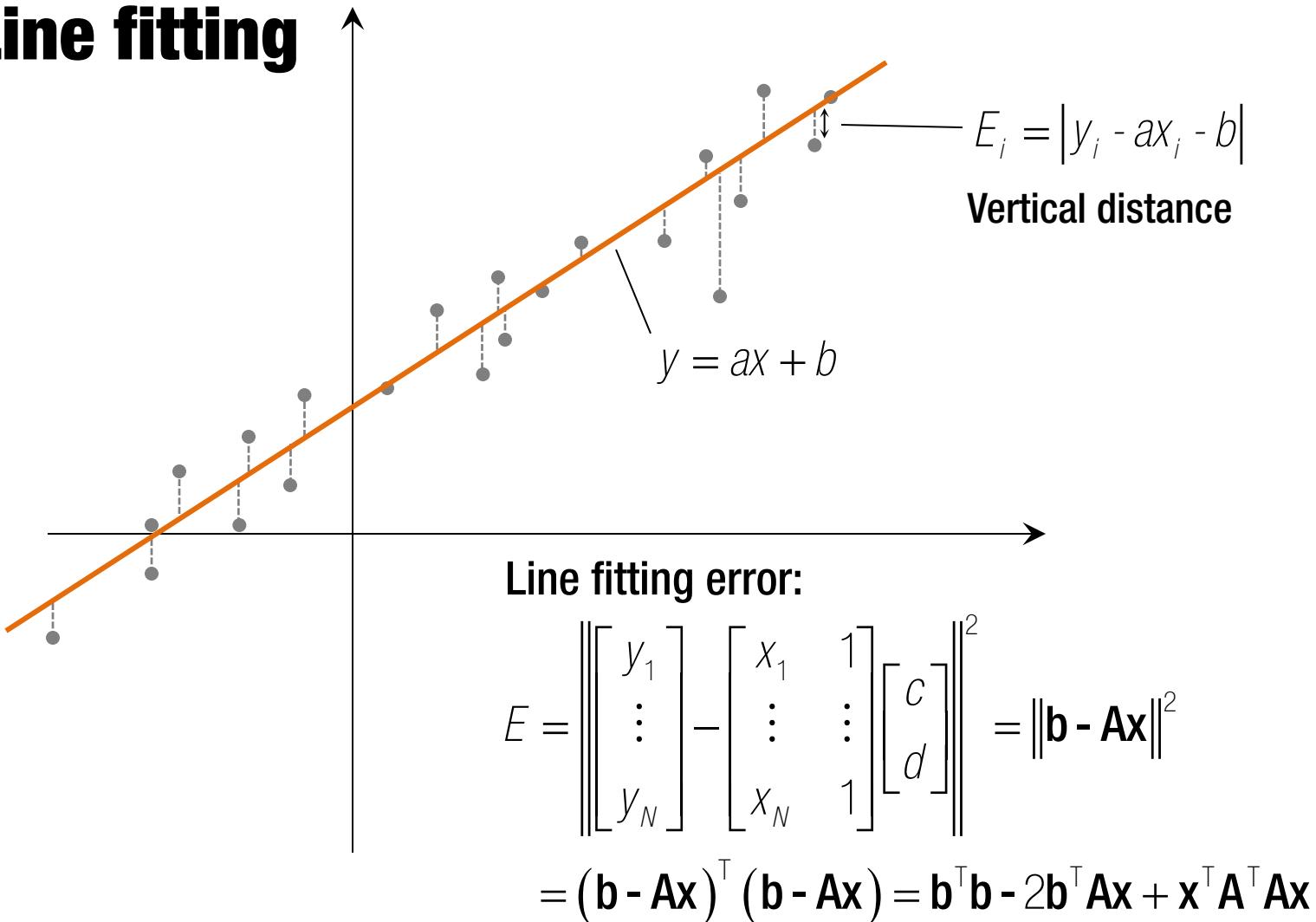
Line fitting



Line fitting

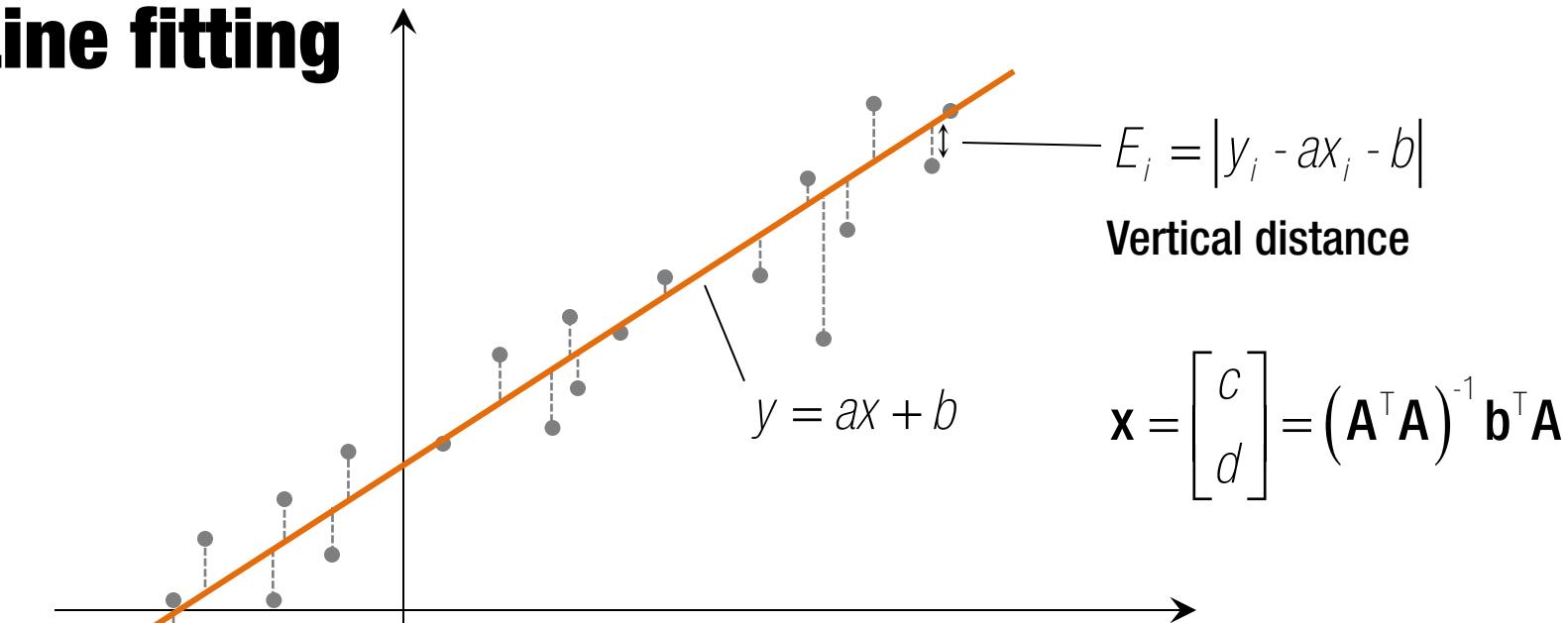


Line fitting



$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^\top \mathbf{A} + 2\mathbf{A}^\top \mathbf{Ax} = \mathbf{0} \rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{b}^\top \mathbf{A}$$

Line fitting



$$\mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{b}^T \mathbf{A}$$

Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^T (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^T \mathbf{b} - 2\mathbf{b}^T \mathbf{Ax} + \mathbf{x}^T \mathbf{A}^T \mathbf{Ax}$$

$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^T \mathbf{A} + 2\mathbf{A}^T \mathbf{Ax} = \mathbf{0} \rightarrow \mathbf{A}^T \mathbf{Ax} = \mathbf{b}^T \mathbf{A}$$

Linear Inhomogeneous Equations

1) $\text{rank}(A) = r < n$: infinite number of solutions

$$x = \underbrace{\mathbf{V} \mathbf{D}^{-1} \mathbf{U}^T \mathbf{b}}_{\text{Particular solution}} + \underbrace{\lambda_{r+1} \mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n}_{\text{Homogeneous solution}}$$

where $A = UDV^T$ and $V = [V_1 \ \dots \ V_n]$.

$$\begin{matrix} A \\ m \times n \end{matrix} \quad \begin{matrix} x \\ n \times 1 \end{matrix} \quad = \quad \begin{matrix} b \\ m \times 1 \end{matrix}$$

Linear Inhomogeneous Equations

1) $\text{rank}(A) = r < n$: infinite number of solutions

$$x = \underbrace{\mathbf{V} \mathbf{D}^{-1} \mathbf{U}^T \mathbf{b}}_{\text{Particular solution}} + \underbrace{\lambda_{r+1} \mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n}_{\text{Homogeneous solution}}$$

Particular solution Homogeneous solution

where $A = UDV^T$ and $V = [\mathbf{V}_1 \ \dots \ \mathbf{V}_n]$.

$$\begin{matrix} A \\ m \times n \end{matrix} \quad \begin{matrix} x \\ n \times 1 \end{matrix} \quad = \quad \begin{matrix} b \\ m \times 1 \end{matrix}$$

2) $\text{rank}(A) = n$: exact solution

$$x = A^{-1}b$$

Linear Inhomogeneous Equations

1) $\text{rank}(A) = r < n$: infinite number of solutions

$$x = \underbrace{\mathbf{V} \mathbf{D}^{-1} \mathbf{U}^T \mathbf{b}}_{\text{Particular solution}} + \underbrace{\lambda_{r+1} \mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n}_{\text{Homogeneous solution}}$$

Particular solution Homogeneous solution

where $A = UDV^T$ and $V = [\mathbf{V}_1 \ \dots \ \mathbf{V}_n]$.

$$\begin{matrix} A \\ m \times n \end{matrix} \quad \begin{matrix} x \\ n \times 1 \end{matrix} \quad = \quad \begin{matrix} b \\ m \times 1 \end{matrix}$$

2) $\text{rank}(A) = n$: exact solution

$$x = A^{-1}b$$

3) $n < m$: no exact solution in general (needs least squares)

$$\min_x \|Ax - b\|^2 \rightarrow x = (A^T A)^{-1} A^T b$$

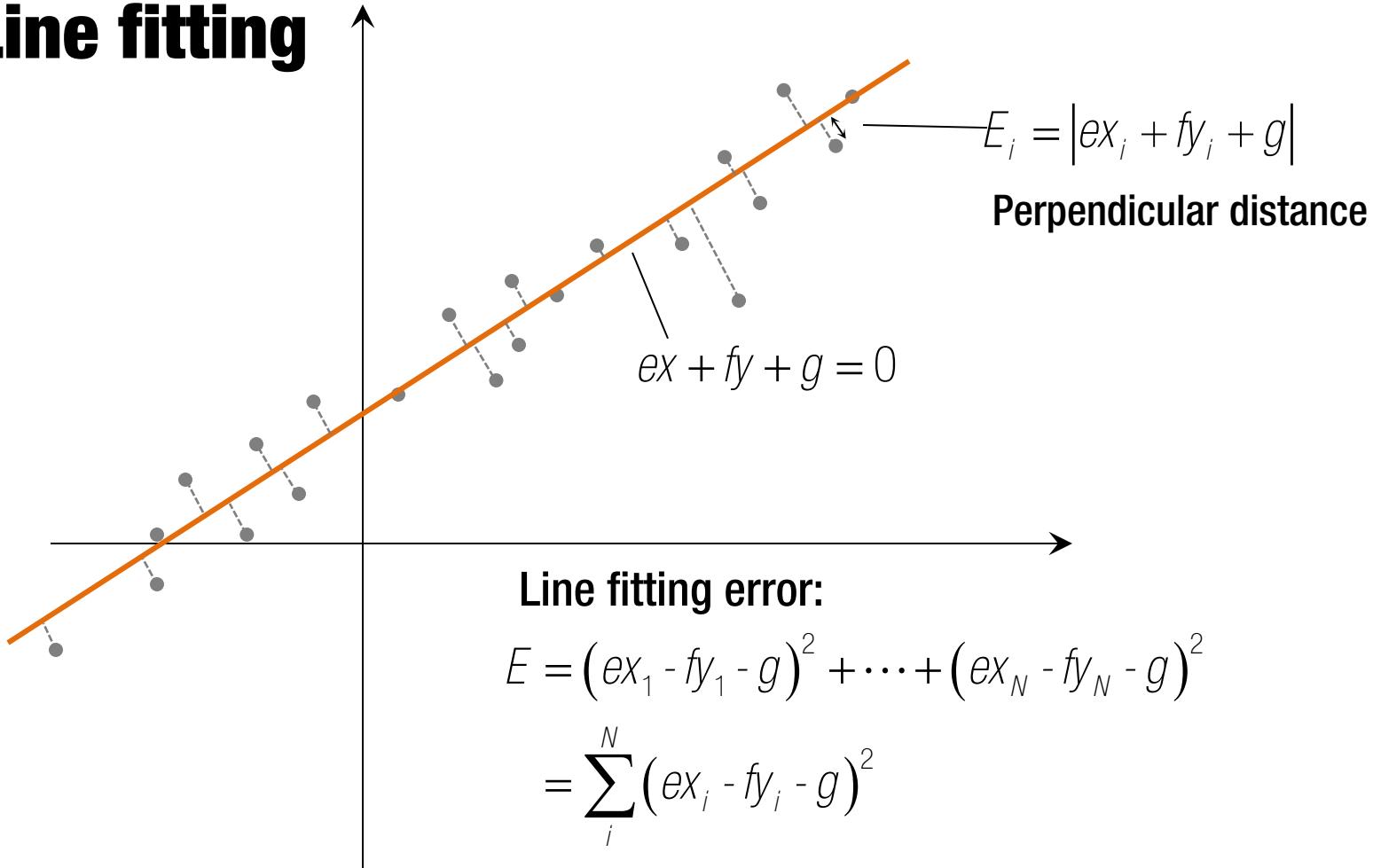
or $x = A \setminus b$ in MATLAB.

Two types of Least Square Problem:

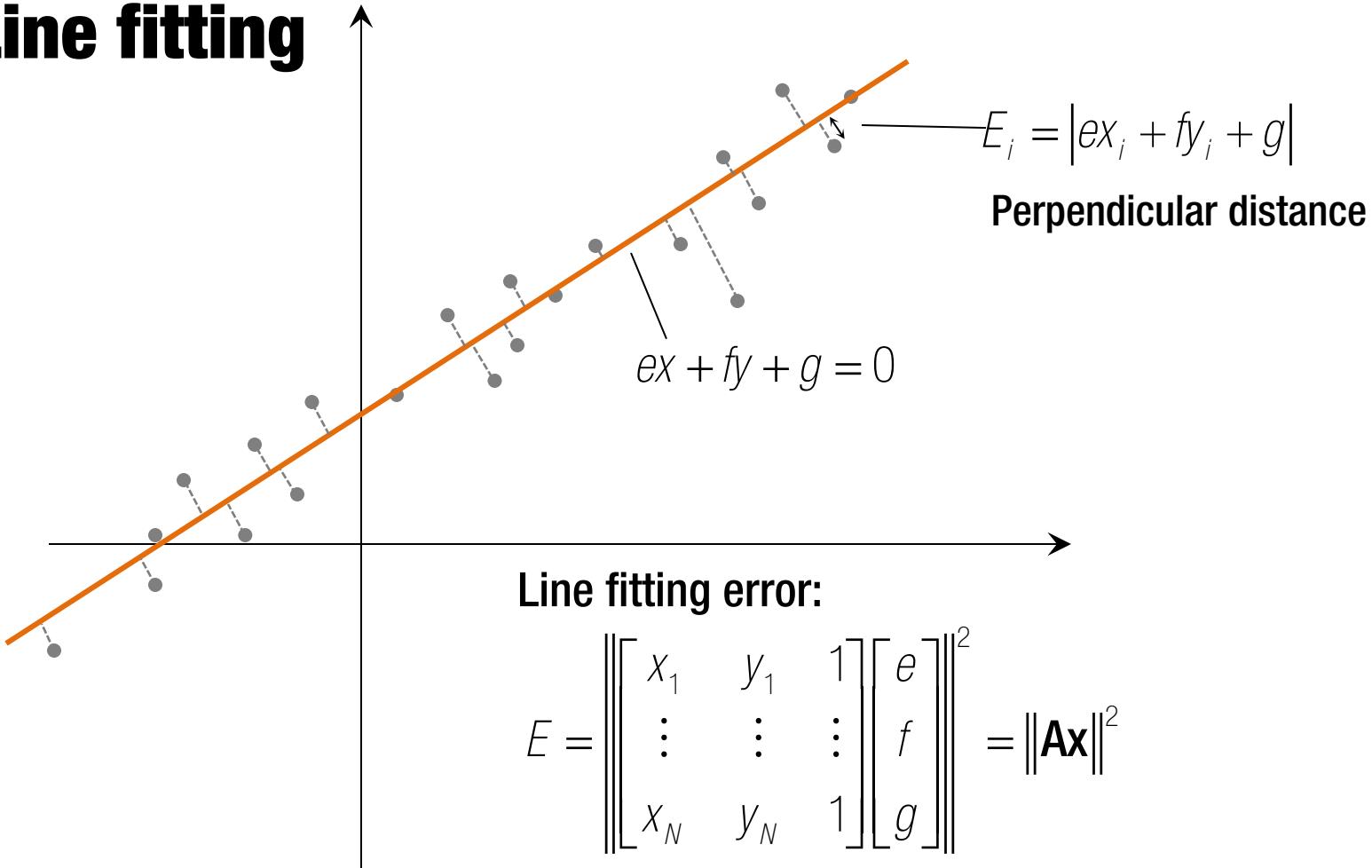
$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|^2 \quad \text{with} \quad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax}\|^2 \quad \text{s.t.} \quad \|\mathbf{x}\| = 1$$

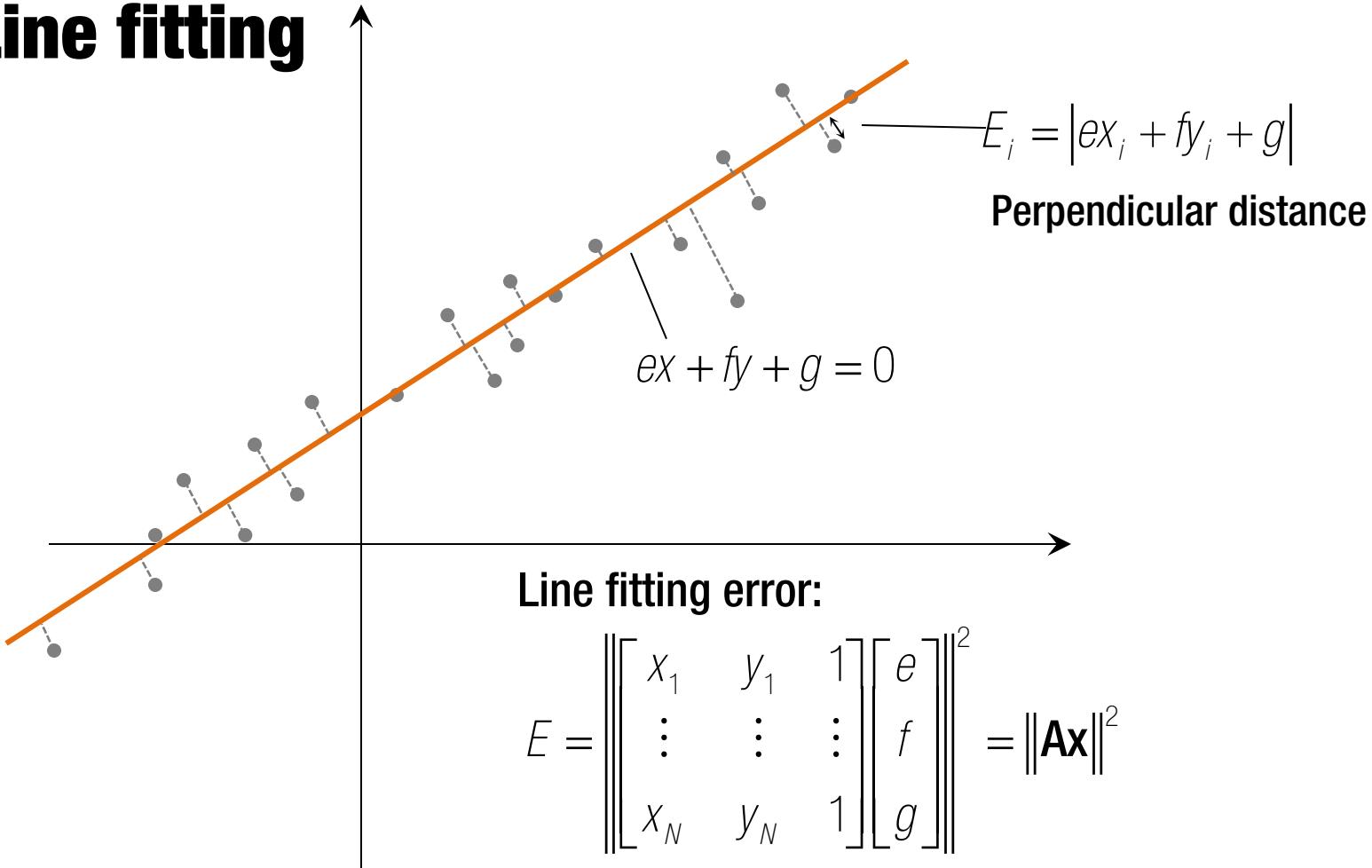
Line fitting



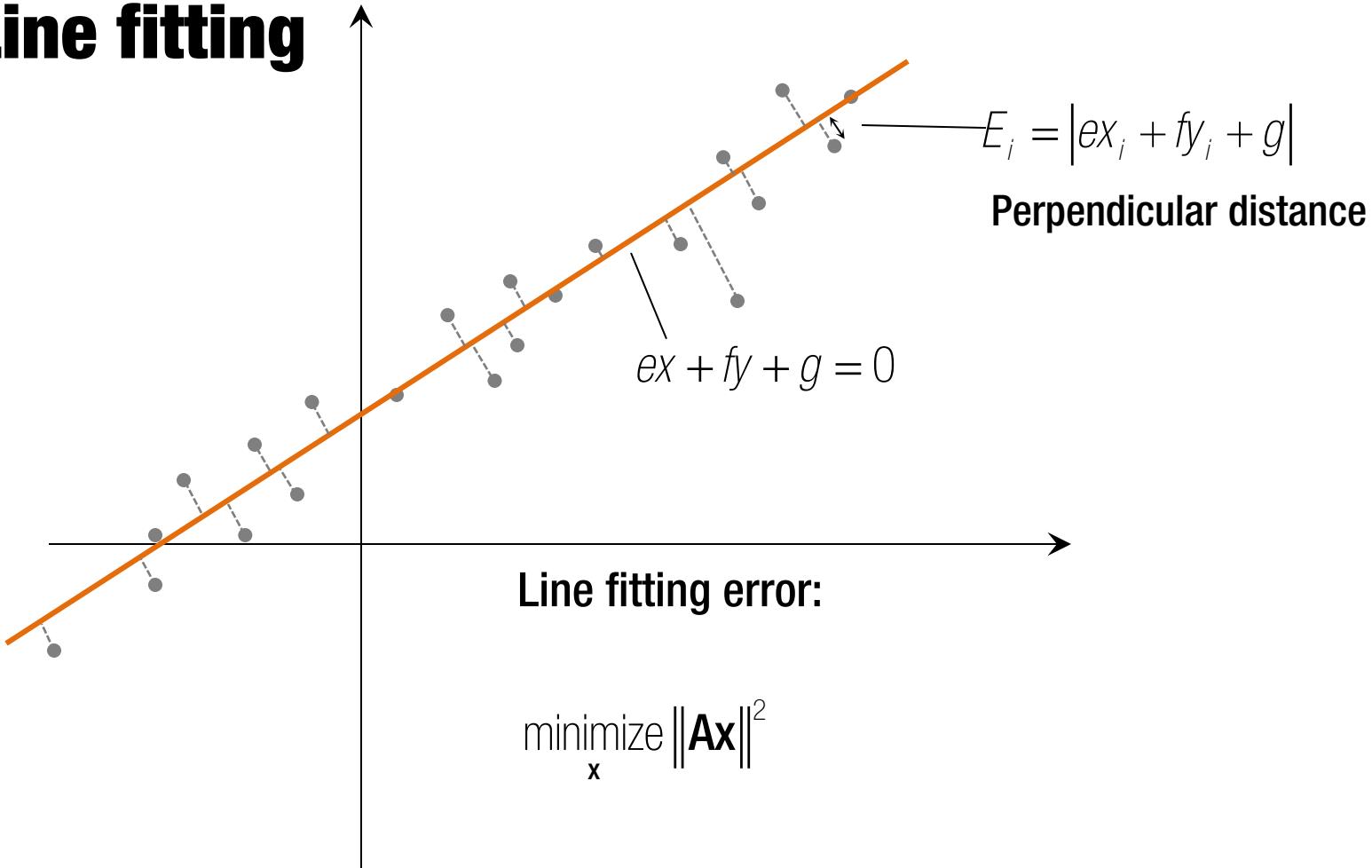
Line fitting



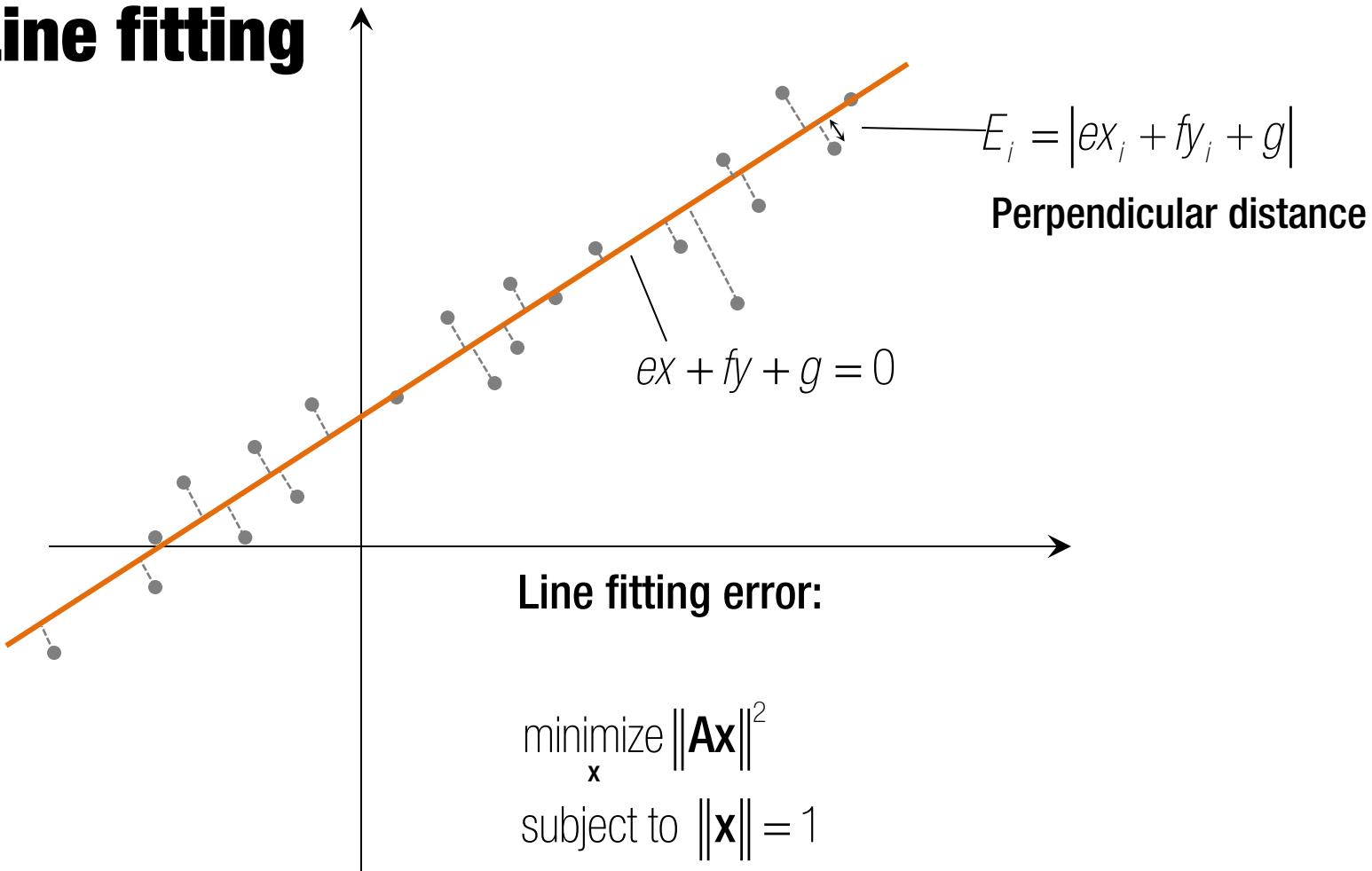
Line fitting



Line fitting



Line fitting



$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

$$\text{subject to } \|\mathbf{x}\| = 1$$

$$\mathbf{x} = \mathbf{V}_3 \quad \text{where } \mathbf{A} = \mathbf{UDV}^T$$
$$\mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2 \quad \mathbf{V}_3]$$

Linear Homogeneous Equations

Linear least square solve produces a trivial solution:

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on \mathbf{X} to avoid the trivial solution: $\|\mathbf{x}\| = 1$

A

x

=

0

$m \times n$

$n \times 1$

$m \times 1$

Linear Homogeneous Equations

Linear least square solve produces a trivial solution:

$$\mathbf{x} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on \mathbf{x} to avoid the trivial solution $\|\mathbf{x}\| = 1$

A

x

=

0

$m \times n$

$n \times 1$

$m \times 1$

1) $\text{rank}(\mathbf{A}) = r < n - 1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1} \mathbf{v}_{r+1} + \cdots + \lambda_n \mathbf{v}_n \quad \text{where} \quad \sum_{i=r+1}^n \lambda_i^2 = 1$$

Linear Homogeneous Equations

Linear least square solve produces a trivial solution:

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on \mathbf{x} to avoid the trivial solution $\|\mathbf{x}\| = 1$

A

x

=

0

$m \times n$

$n \times 1$

$m \times 1$

1) $\text{rank}(\mathbf{A}) = r < n - 1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1} \mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n \quad \text{where} \quad \sum_{i=r+1}^n \lambda_i^2 = 1$$

2) $\text{rank}(\mathbf{A}) = n - 1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

Linear Homogeneous Equations

Linear least square solve produces a trivial solution:

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on \mathbf{x} to avoid the trivial solution $\|\mathbf{x}\| = 1$

$$\begin{matrix} \mathbf{A} & \mathbf{x} & = & \mathbf{0} \\ m \times n & n \times 1 & & m \times 1 \end{matrix}$$

1) $\text{rank}(\mathbf{A}) = r < n - 1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1} \mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n \quad \text{where} \quad \sum_{i=r+1}^n \lambda_i^2 = 1$$

2) $\text{rank}(\mathbf{A}) = n - 1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

3) $n < m$: no exact solution in general (needs least square)

$$\min_{\mathbf{x}} \|\mathbf{Ax}\|^2 \quad \text{subject to} \|\mathbf{x}\| = 1 \rightarrow \mathbf{x} = \mathbf{V}_n$$