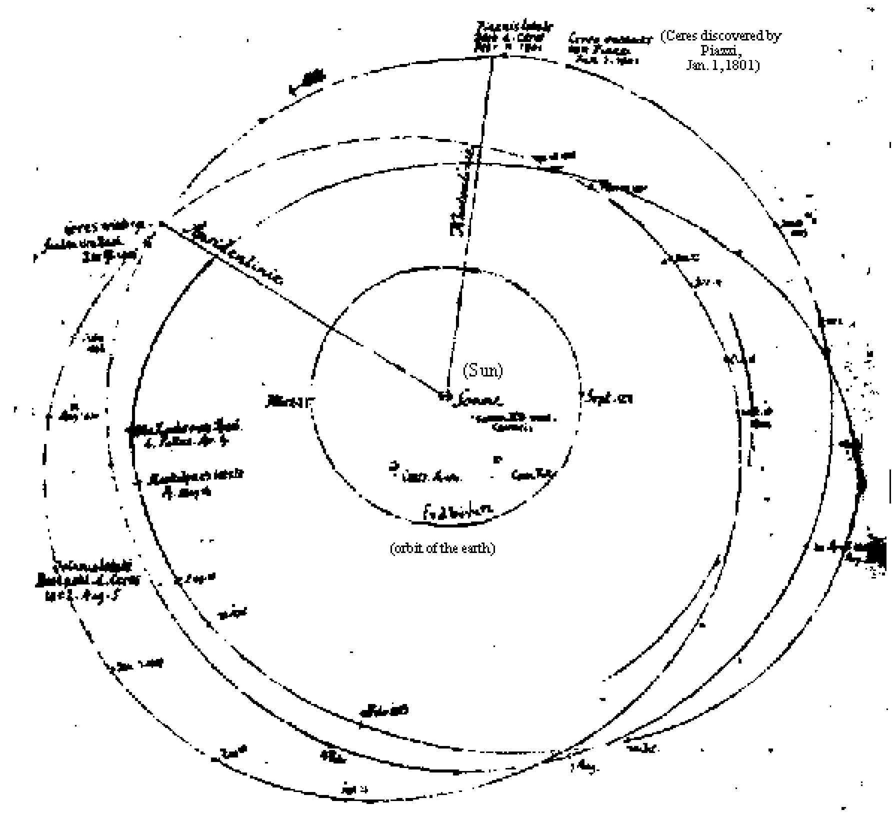
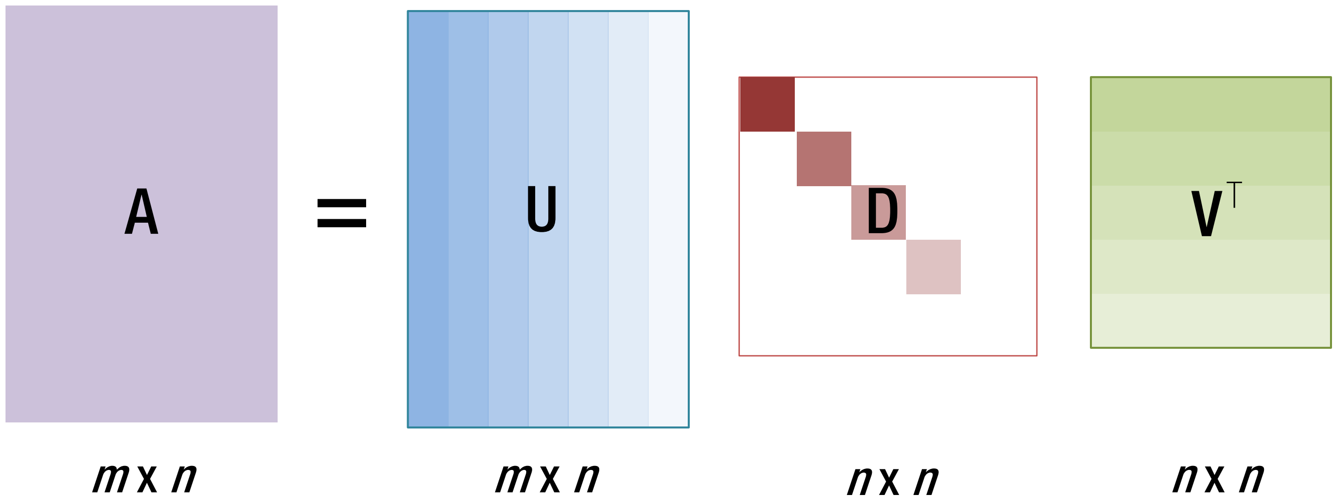
1809, Carl Friedrich Gauss

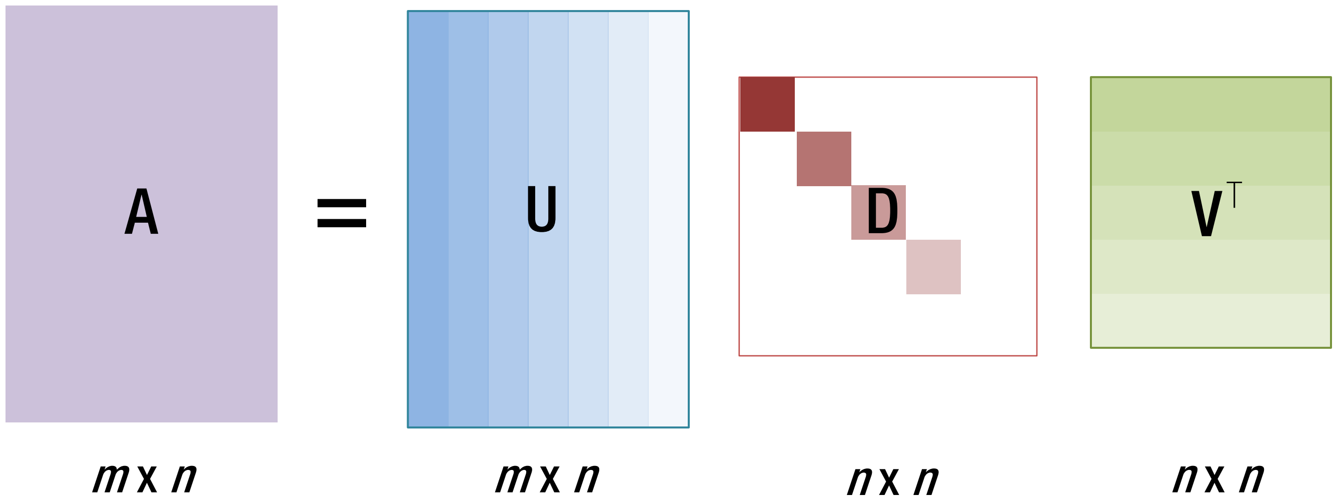


Sketch of the orbits of Ceres and Pallas (nachlaß Gauß, Handb. 4). Courtesy of Universitätsbibliothek Göttingen.

Singular Value Decomposition



Singular Value Decomposition

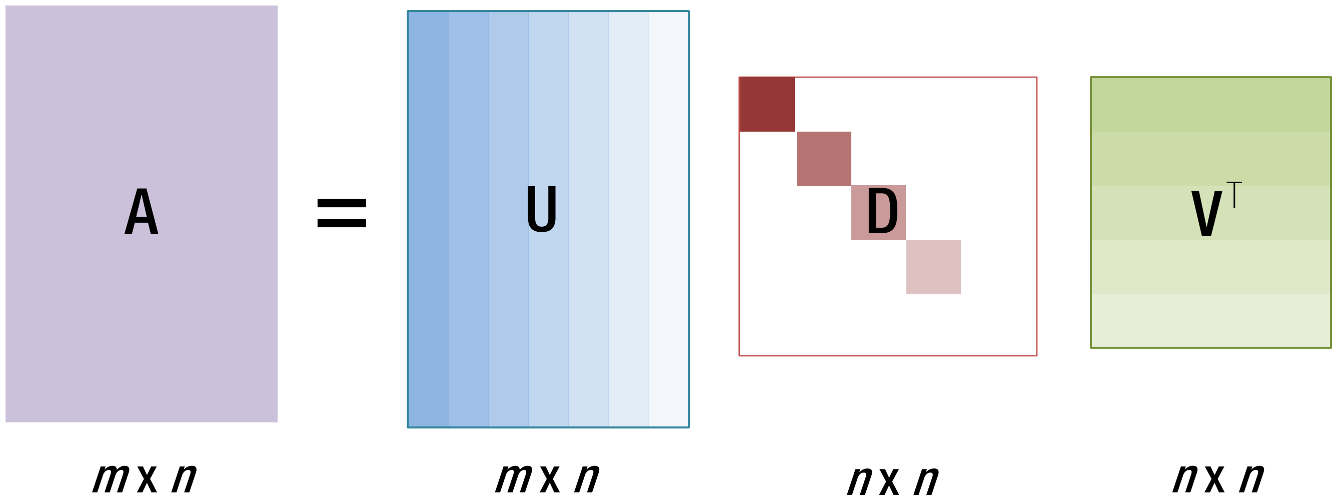


Column orthogonal matrix

$$\mathbf{U}_i^\top \mathbf{U}_i = \|\mathbf{U}_i\| = 1$$

$$\mathbf{U}_j^\top \mathbf{U}_i = \mathbf{U}_j^\top \mathbf{U}_i = 0 \quad \text{for} \quad i \neq j$$

Singular Value Decomposition



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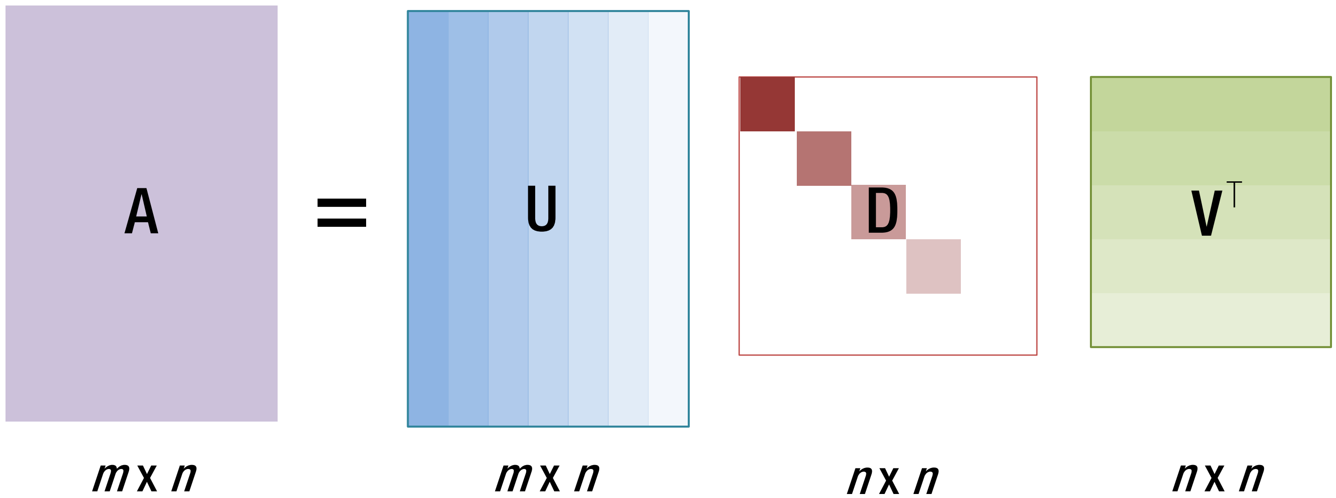
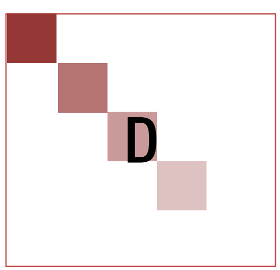
$$\mathbf{U}^{\top}\mathbf{U} = \mathbf{I}_{m\times m}$$

$$\mathbf{V}^{\top}\mathbf{V} = \mathbf{I}_{n\times n}$$

Singular Value Decomposition

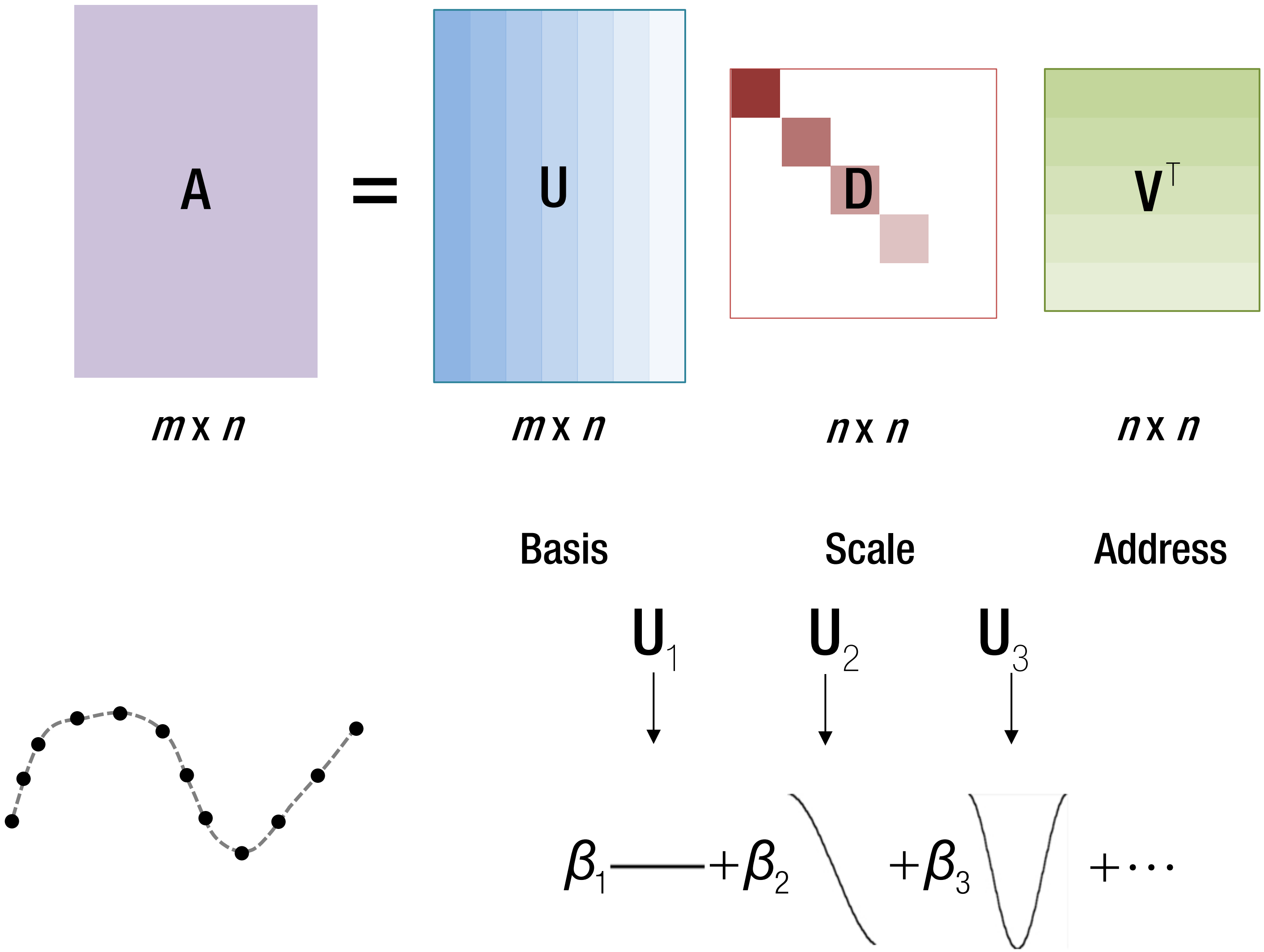


Singular value matrix

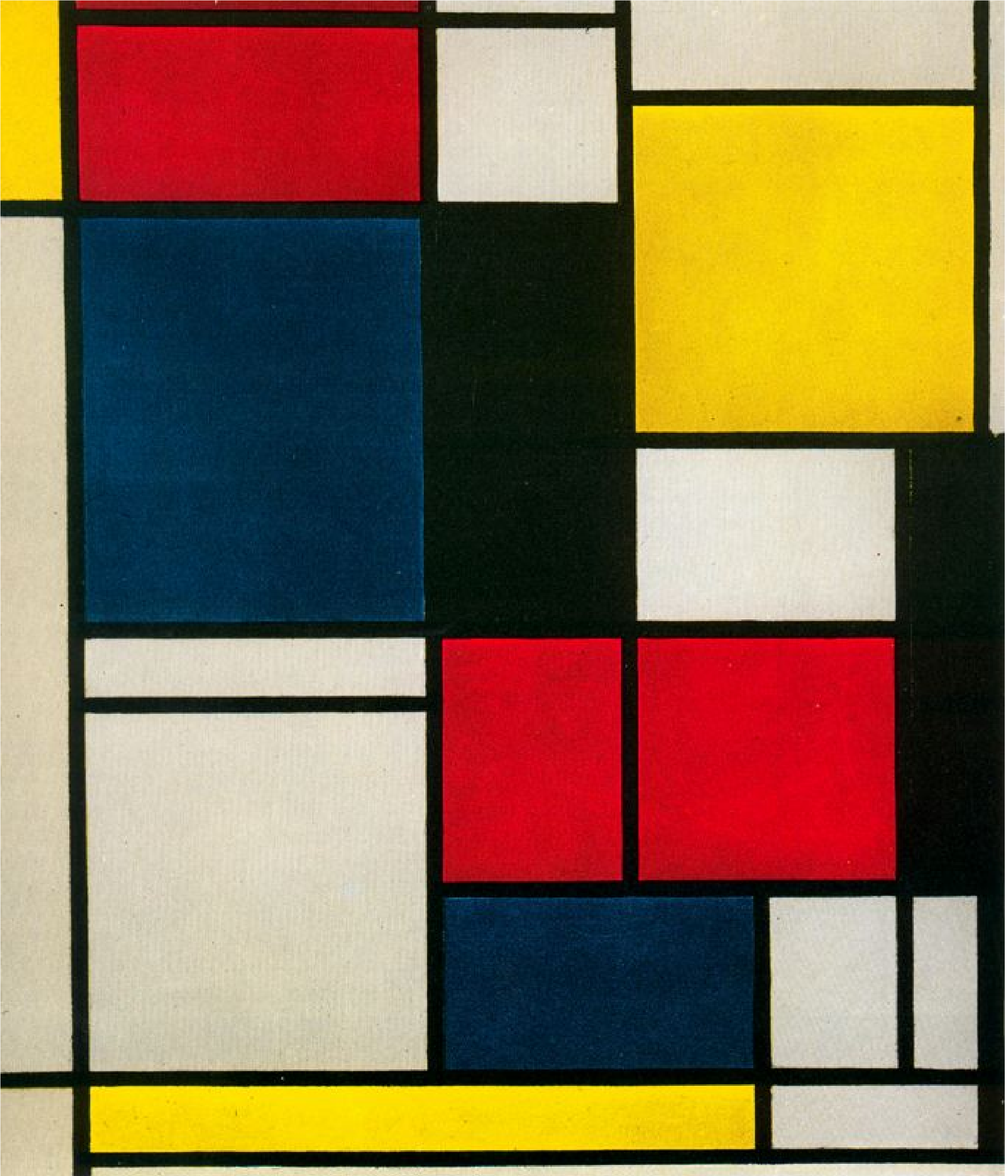


$= \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

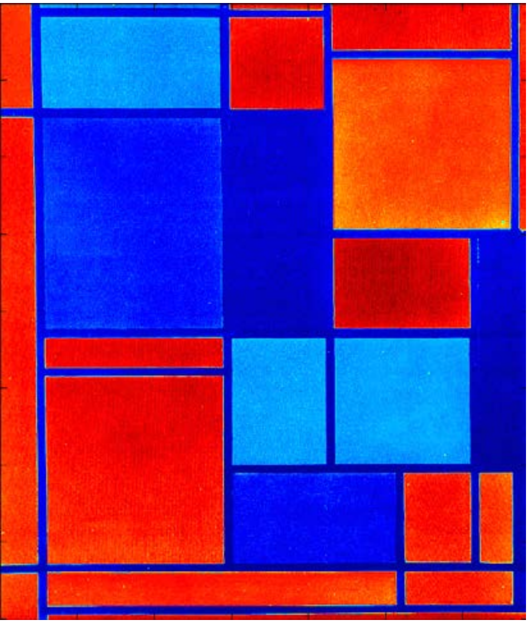
Singular Value Decomposition



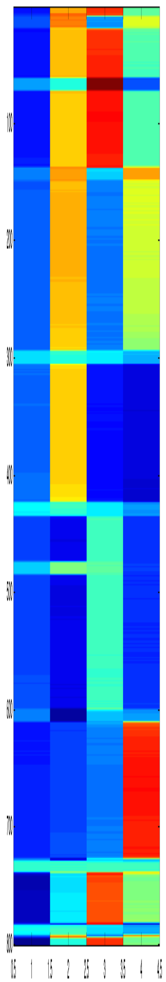
SVD as basis + transformed Address

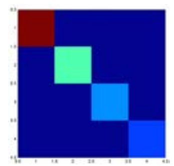


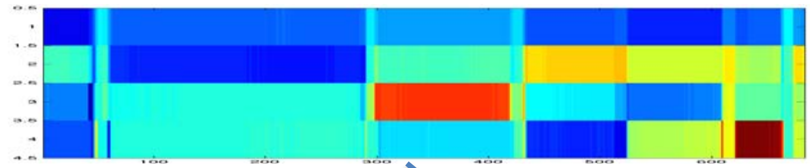
SVD of this?

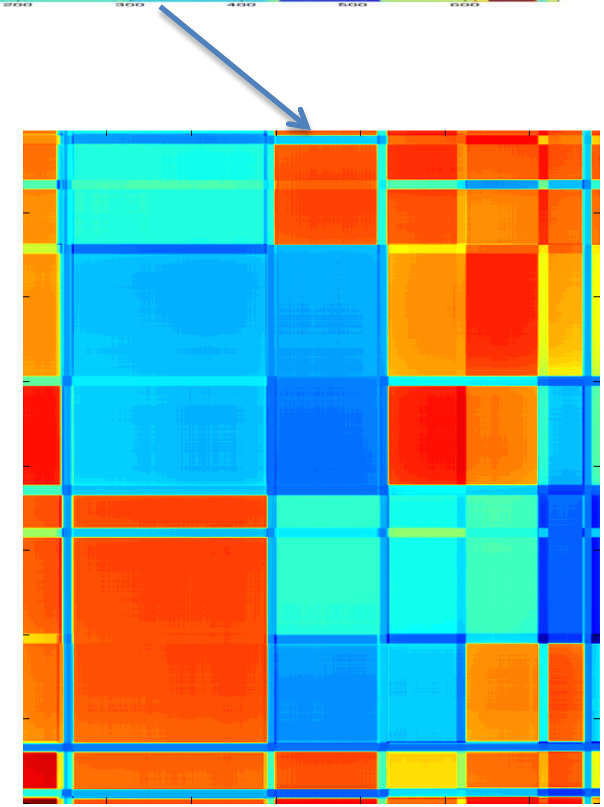


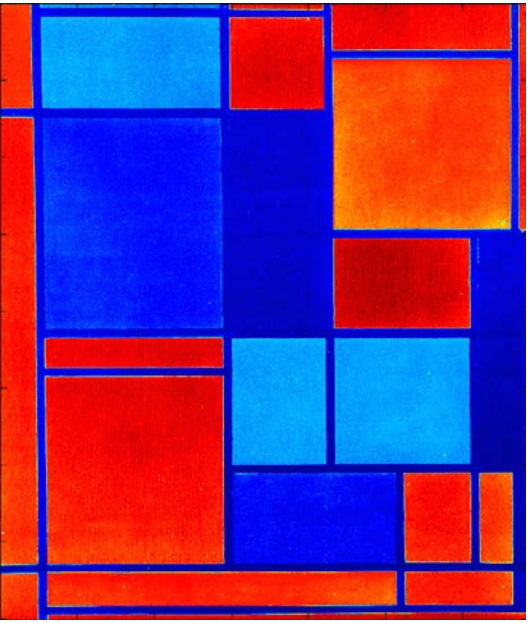
U(:,1:4) D(1:4,1:4) V(:,1:4)’

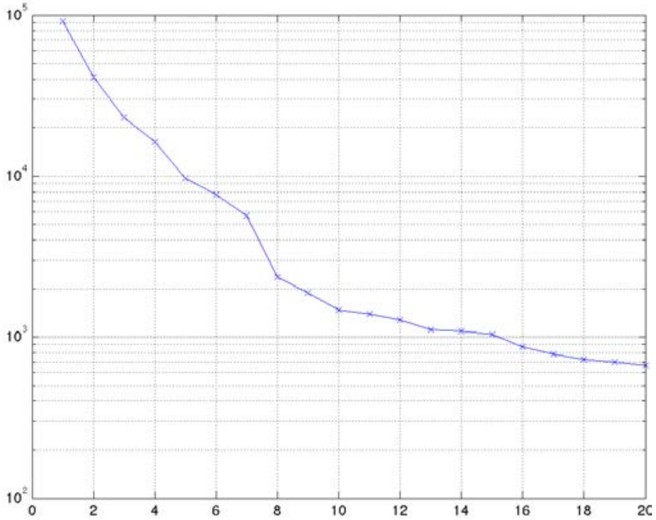


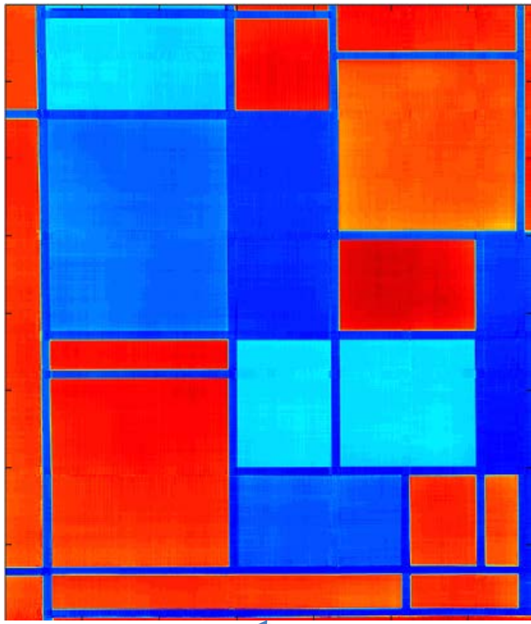








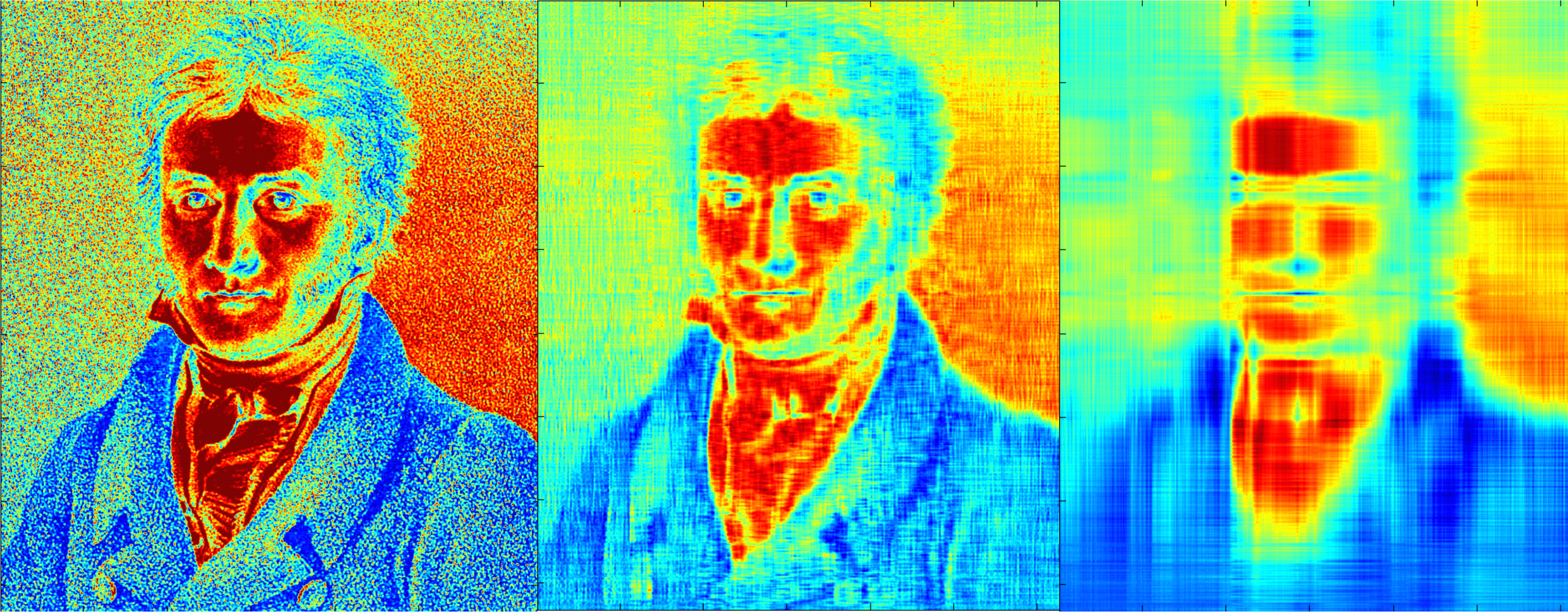


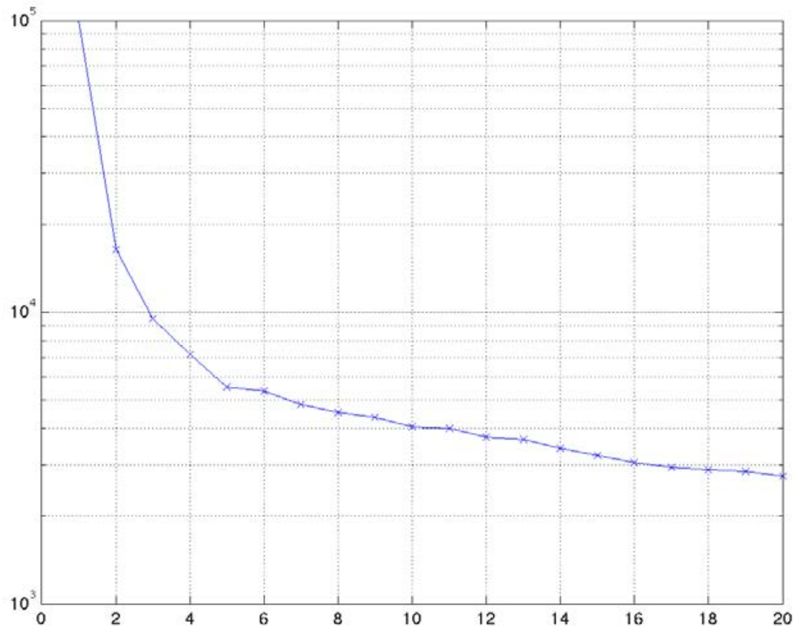


```
[u,d,v] = svd(I);
```

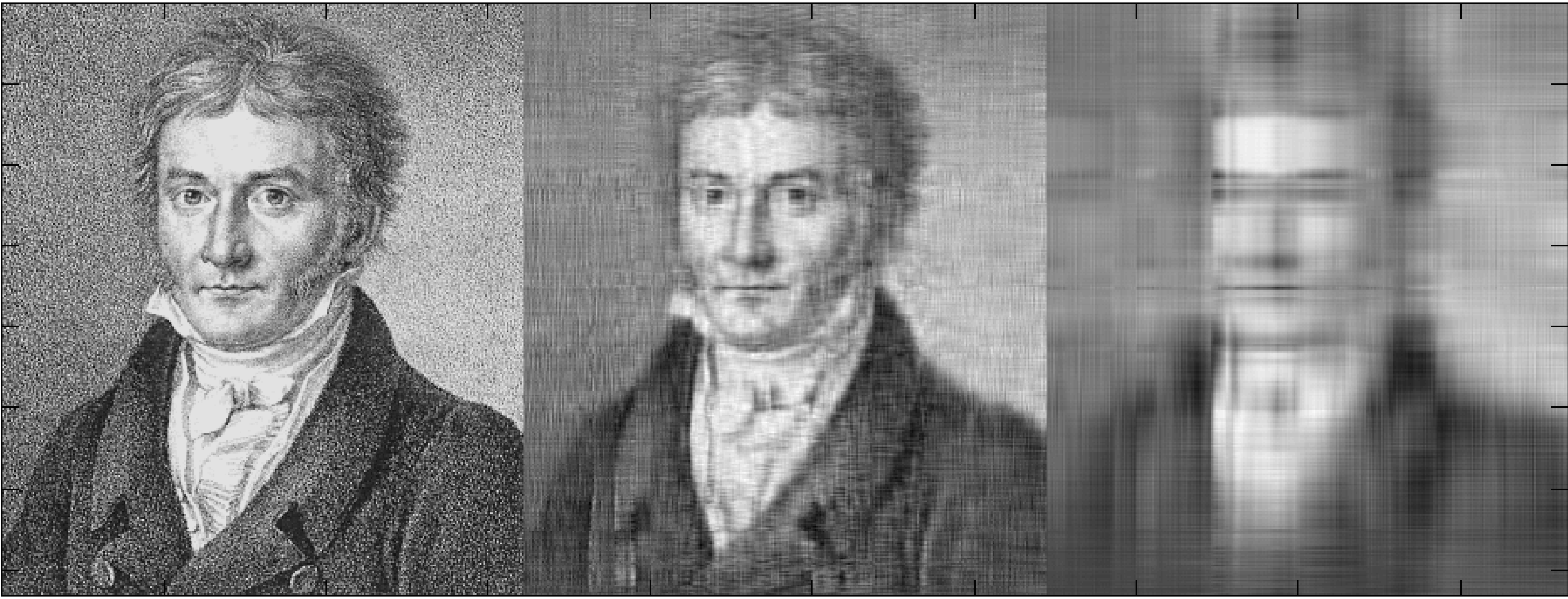
```
semilogy(diag(d(1:20,1:20)),'x-')
```

```
Im2 = u(:,1:20)*d(1:20,1:20)*v(:,1:20)';
```

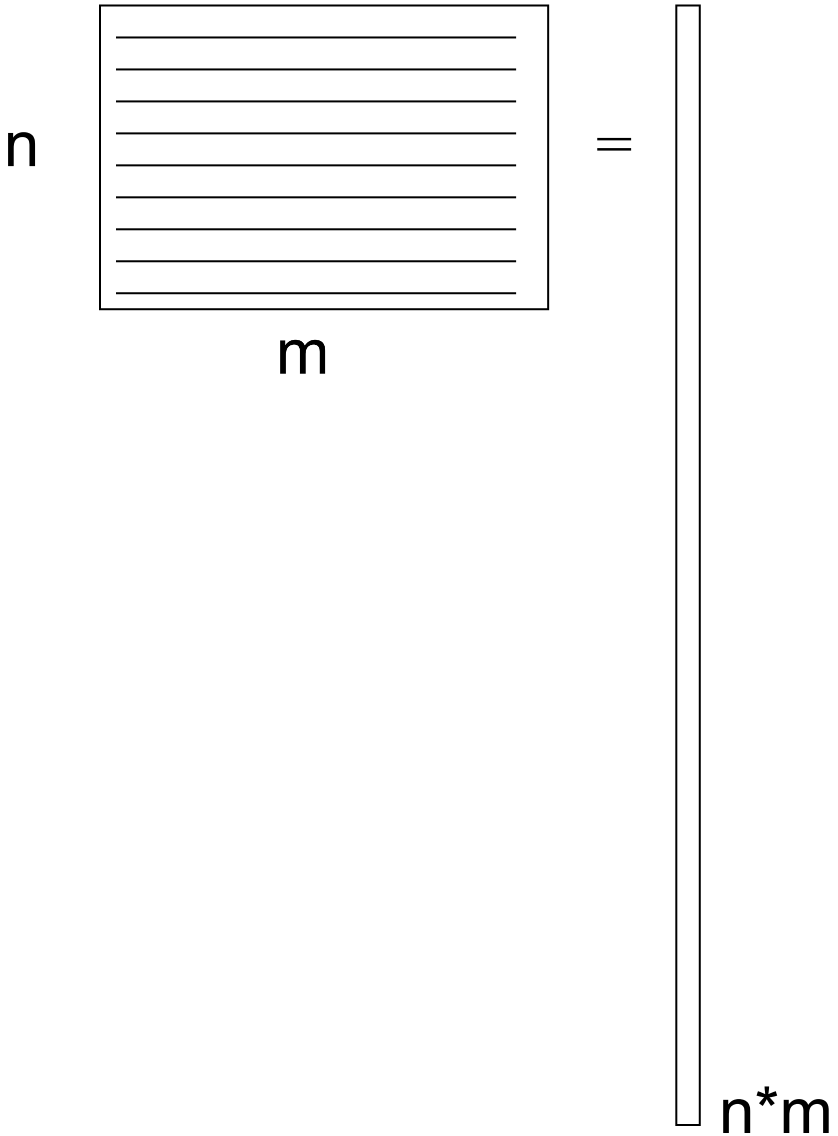




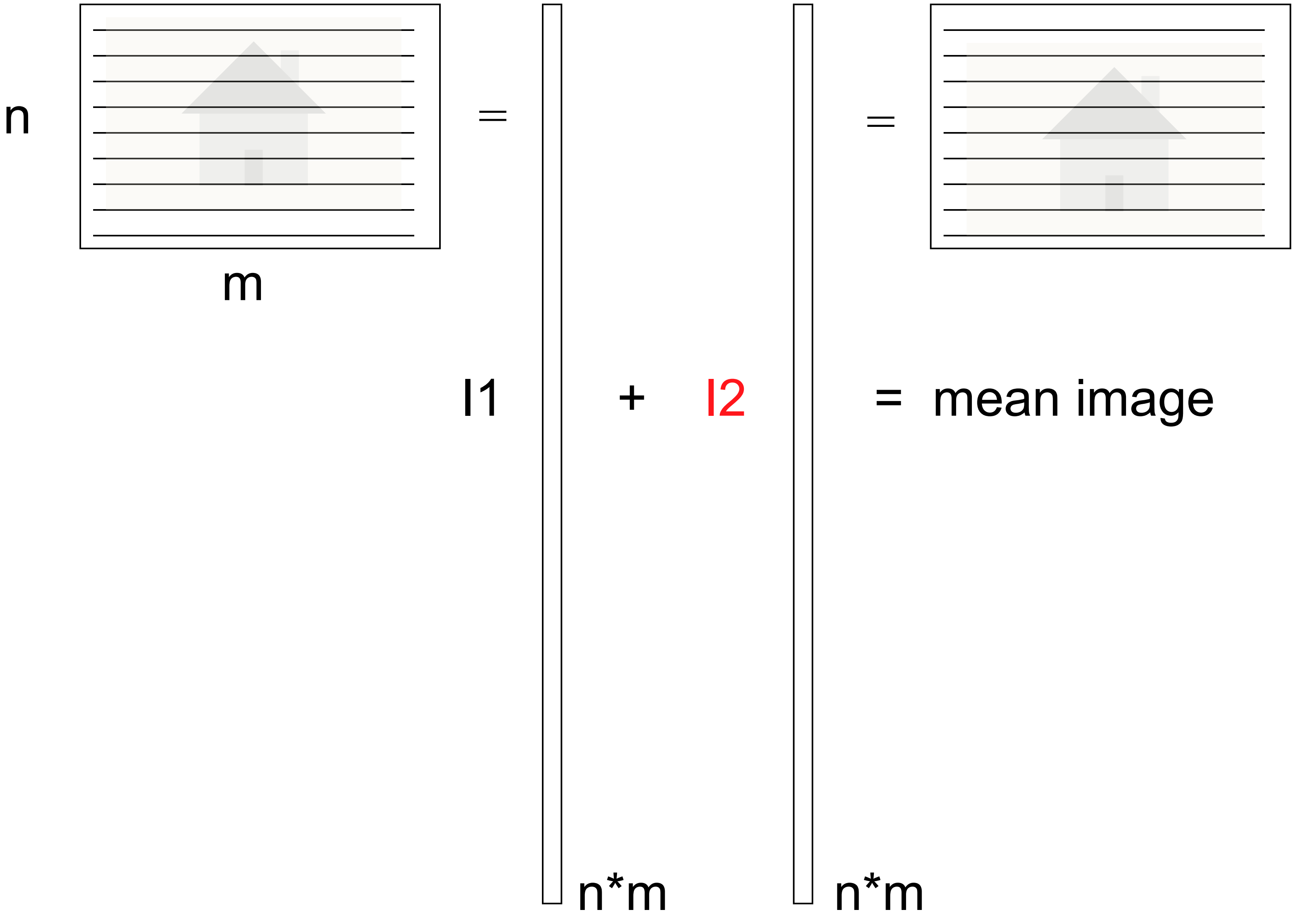
10^5
10^4
10^3
0
2
4
6
8
10
12
14
16
18
20



Images as Vectors



Vector Mean



Eigenfaces

Eigenfaces look somewhat like generic faces.

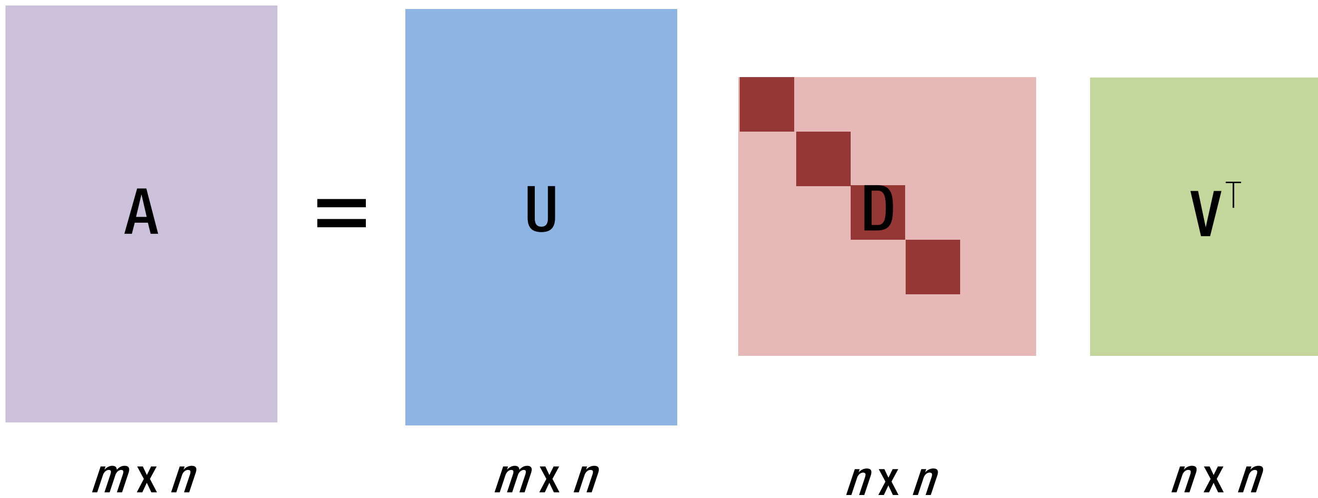
Eigen-images of Berlin

Eigen-images

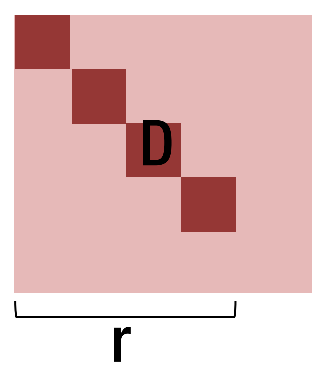
Average of 16 individuals transformed via biometrical data of different ethnics

Average of 16 individuals transformed via
biometrical data of different ages

Rank

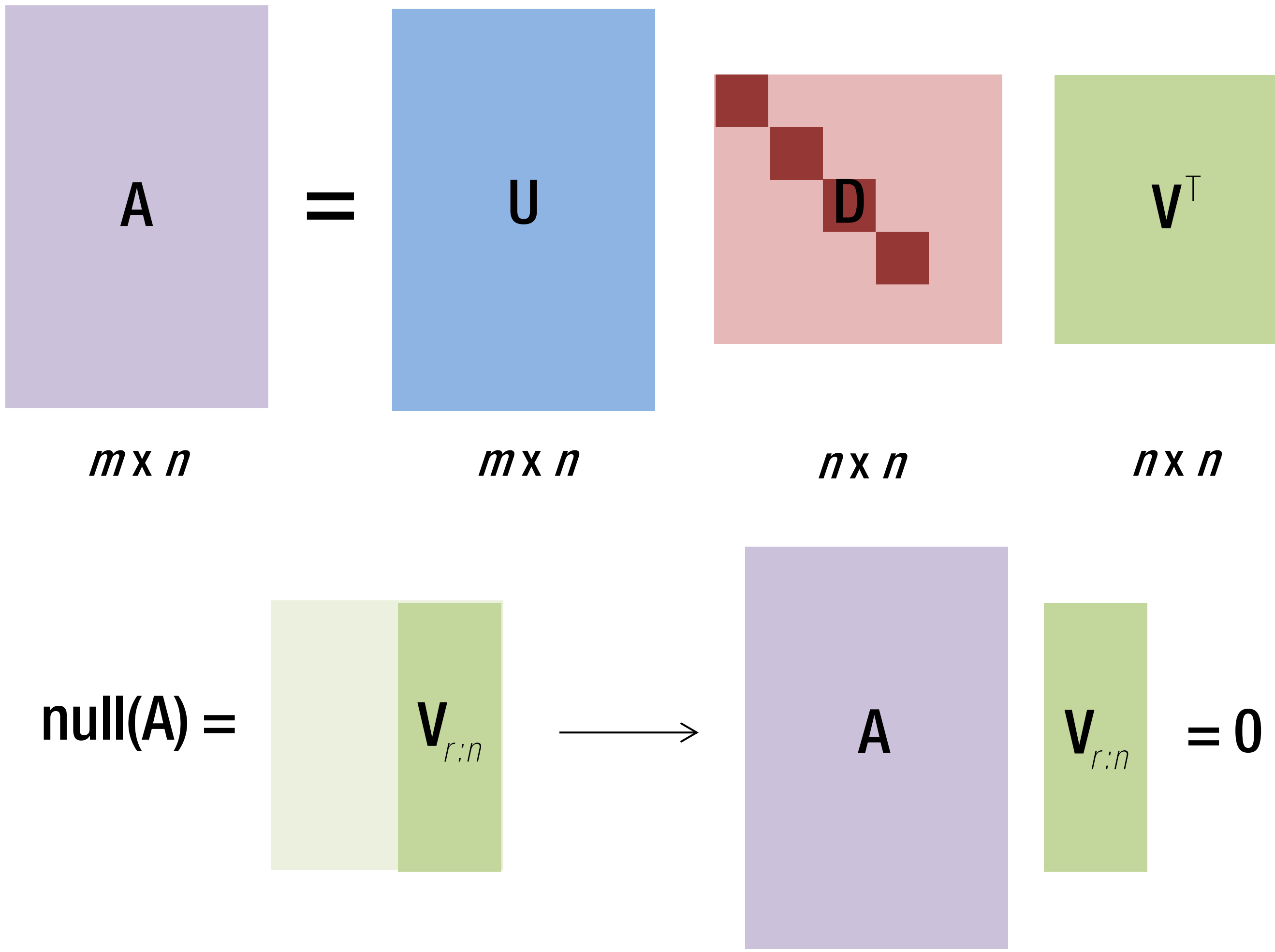


$$\mathrm{rank}(A) = r \leq \min(m, n)$$



Rank is the same as the number of nonzero singular values

Nullspace



Example III (Fundamental Matrix)

F =
 1.0e+003 *

-0.0000	0.0000	0.0030
-0.0001	0.0002	0.0564
0.0132	-0.0292	-9.9998

[u,d,v] = svd(F)

u =

-0.0003	0.9981	0.0618
-0.0056	-0.0618	0.9981
1.0000	-0.0001	0.0056

v =

0.0013	-0.9660	0.2586
-0.0029	-0.2586	-0.9660
-1.0000	-0.0005	0.0032

d =
 1.0e+004 *

1.0000	0	0
0	0.0000	0
0	0	0.0000

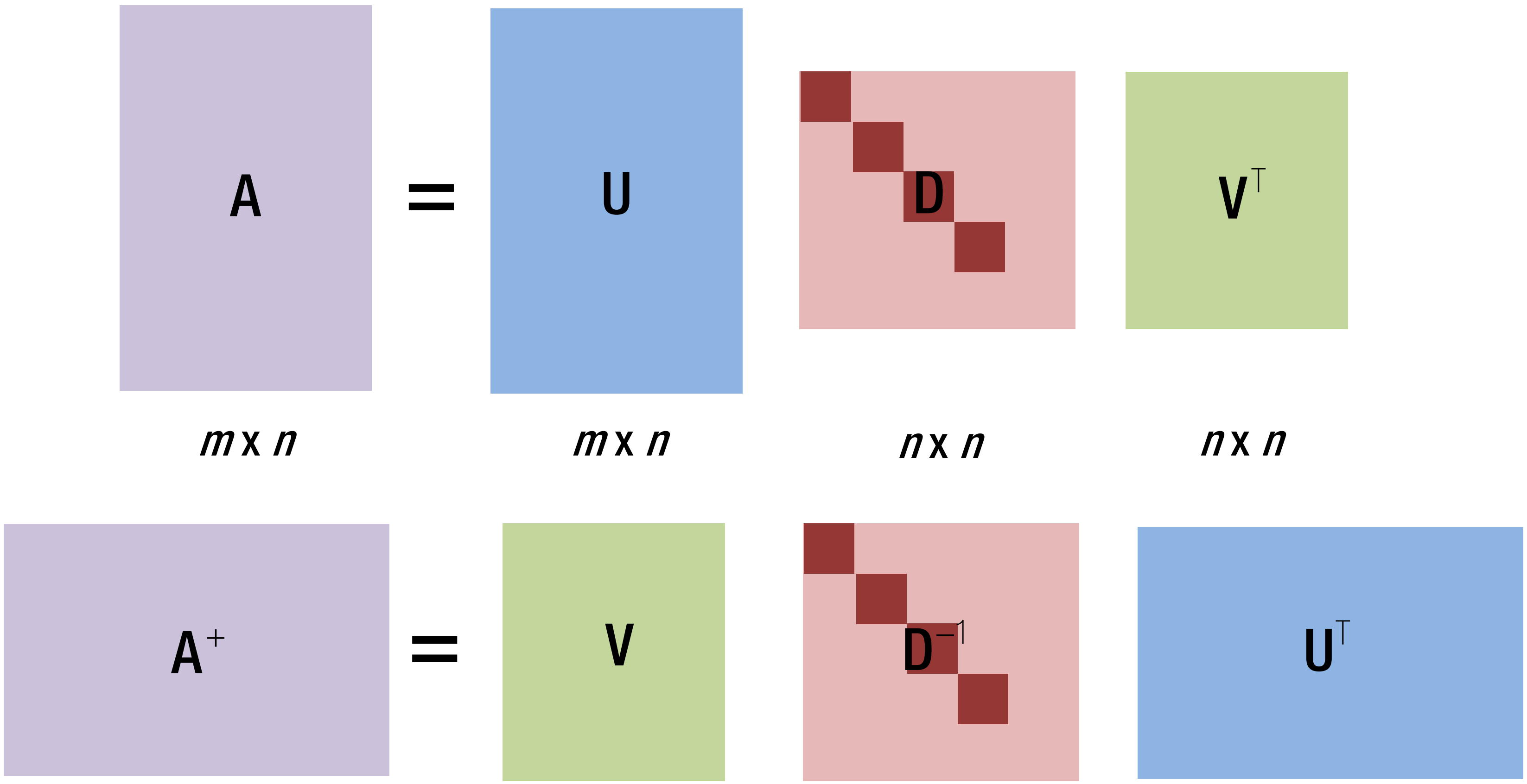
Rank(F) = 2

d(1,1)
ans = 1.0000e+004

d(2,2)
ans = 0.0021

d(3,3)
ans = 2.7838e-016

Matrix Inversion with SVD



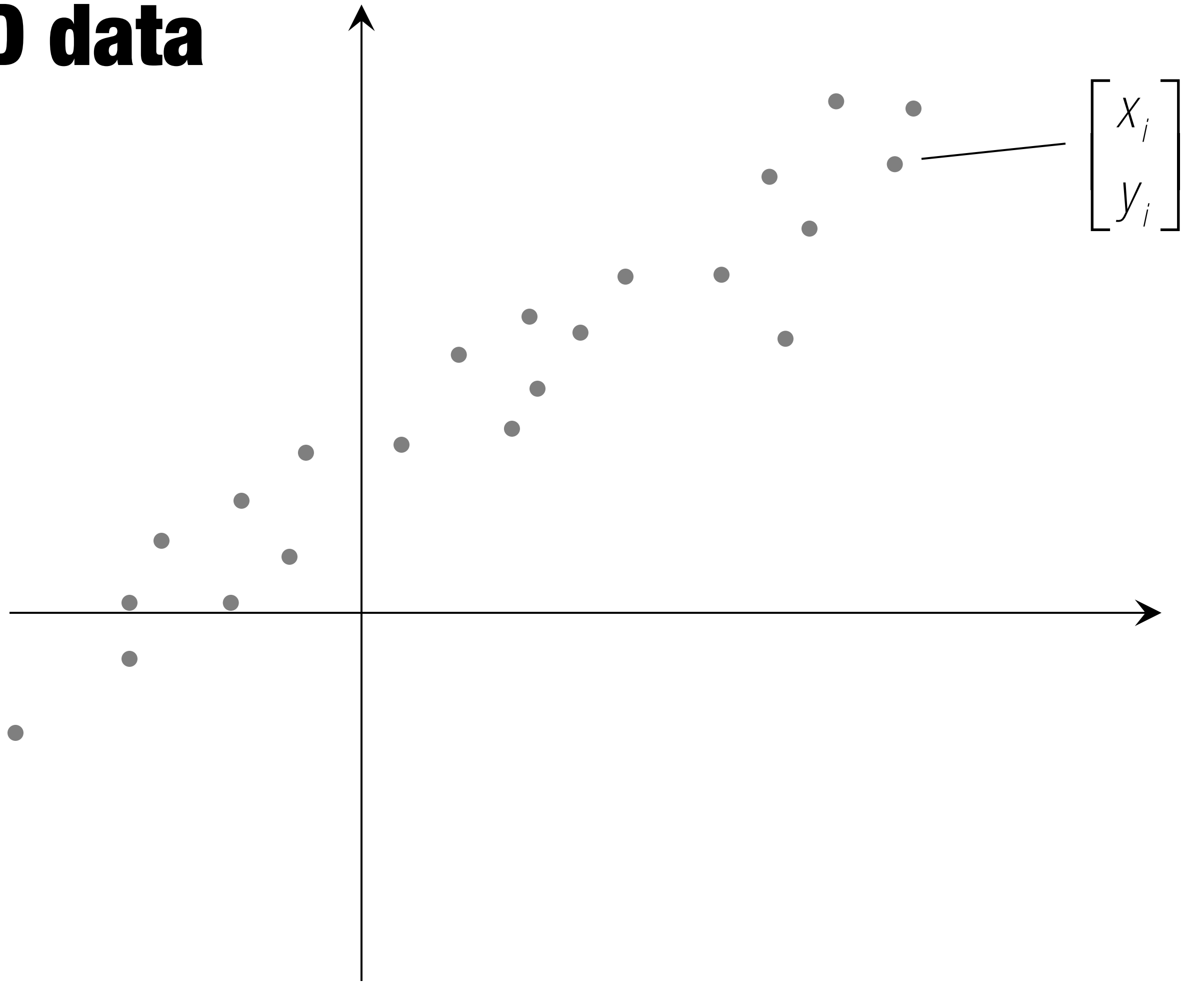
$\mathbf{A}^{+}\mathbf{A} = \mathbf{I}$ $\quad \mathbf{D}^{-1} = \mathrm{diag}\{1/\sigma_1, 1/\sigma_2, \cdots, 1/\sigma_n\}$ **if** $\sigma_i > 0$ **, otherwise zero.**

Two types of Least Square Problem:

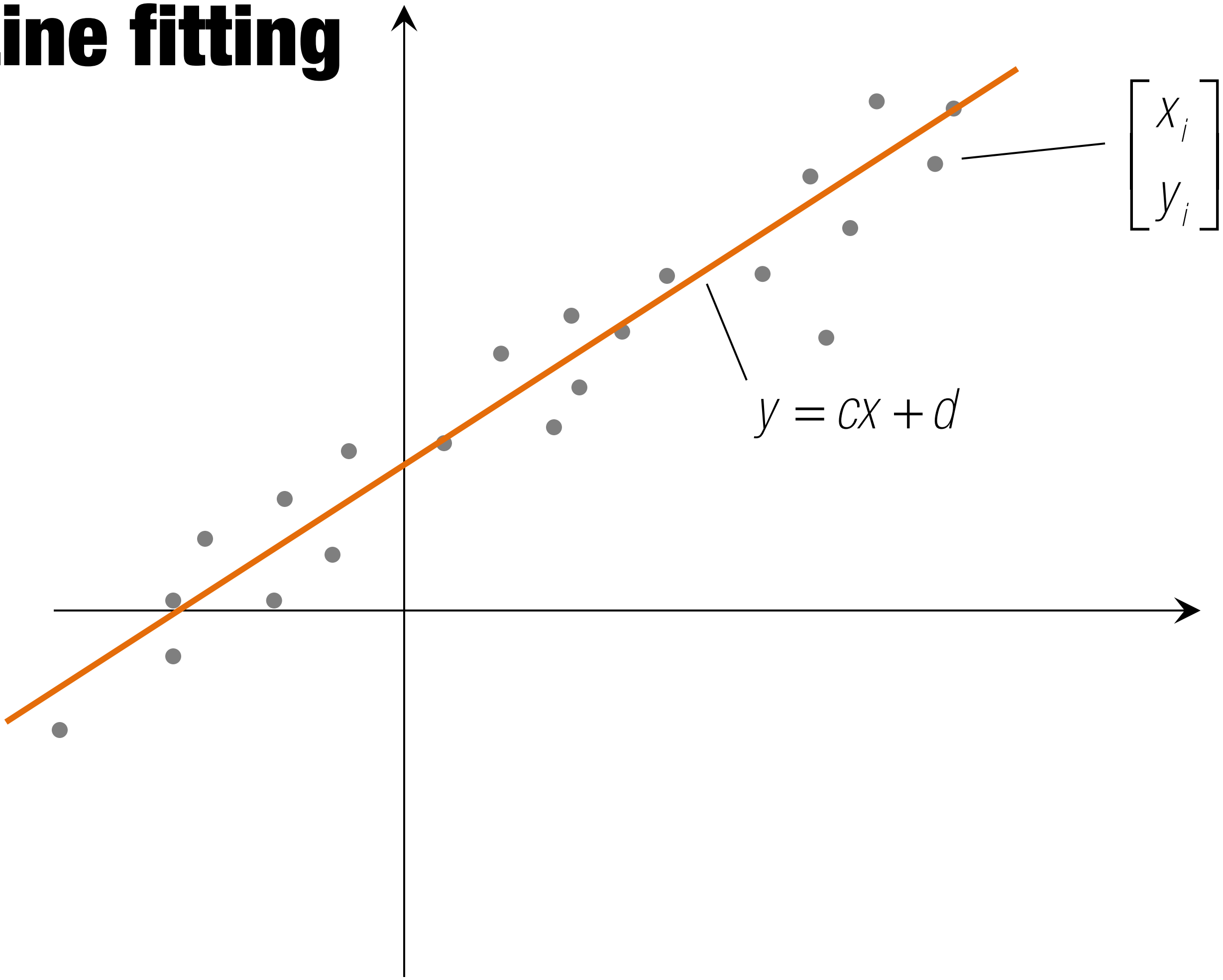
$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x} - \mathbf{b}\|^2 \qquad \text{with} \qquad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathsf{A}\mathbf{x}\|^2 \qquad \text{s.t.} \qquad \|\mathbf{x}\| = 1$$

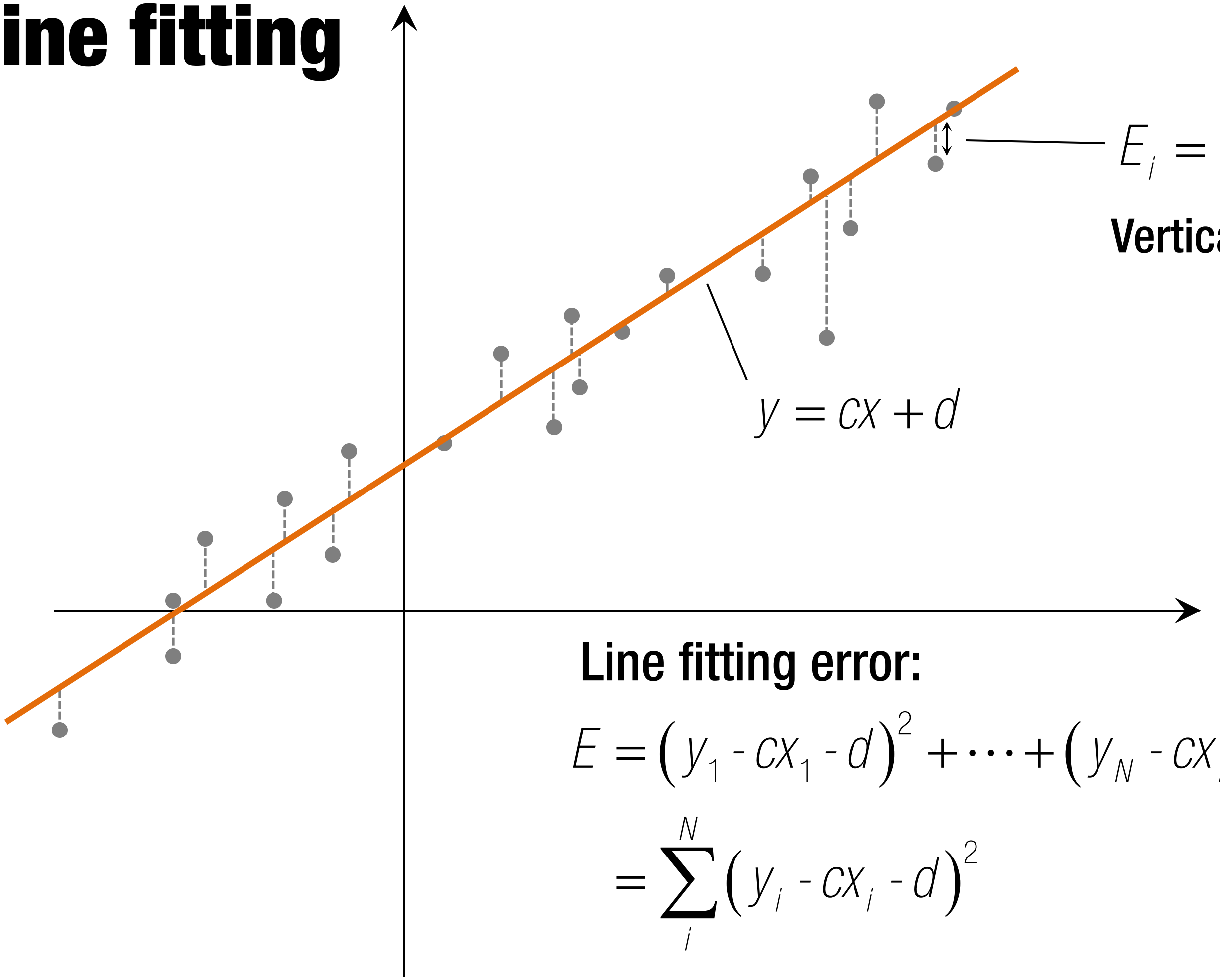
2D data



Line fitting



Line fitting



$$E_i = \left| y_i - cx_i - d \right|$$

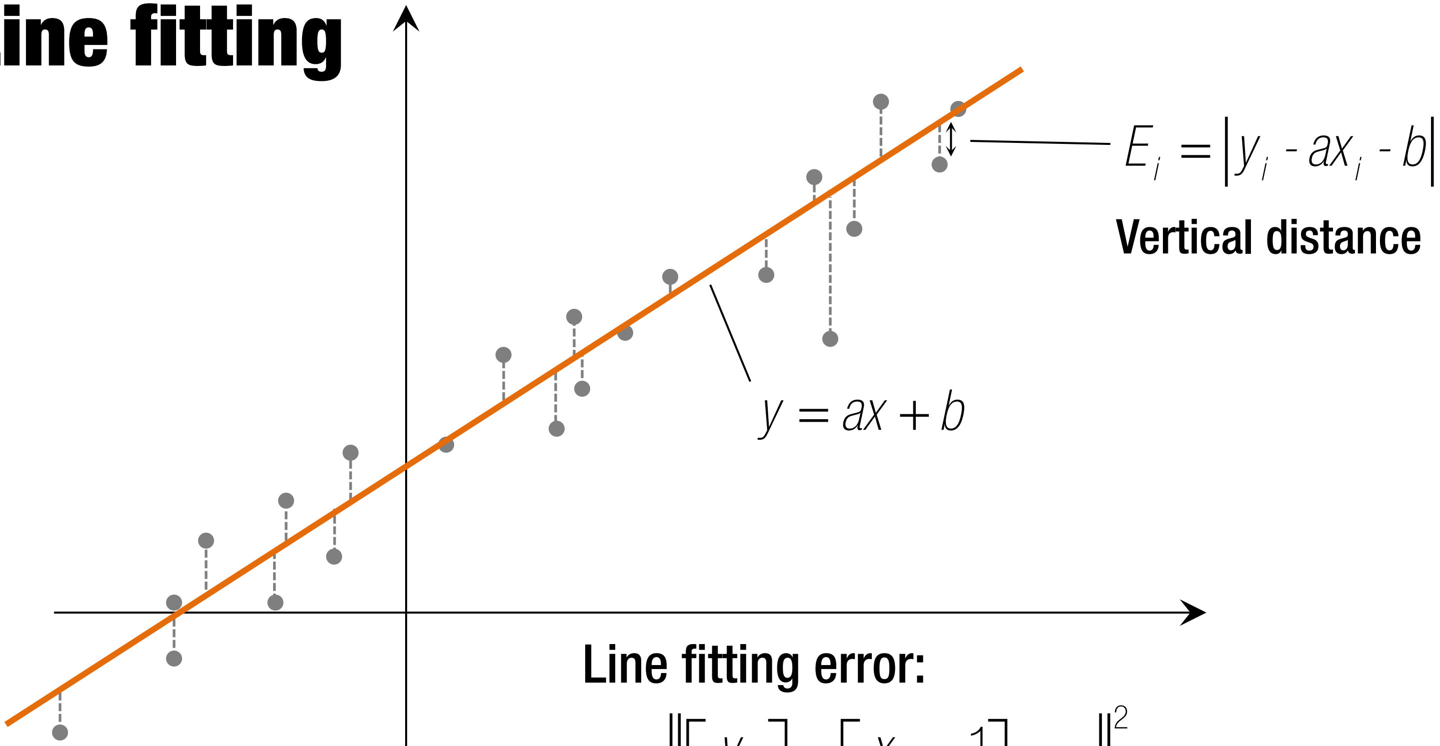
Vertical distance

Line fitting error:

$$E = \left(y_1 - cx_1 - d\right)^2 + \cdots + \left(y_N - cx_N - d\right)^2$$

$$= \sum_i^N \left(y_i - cx_i - d\right)^2$$

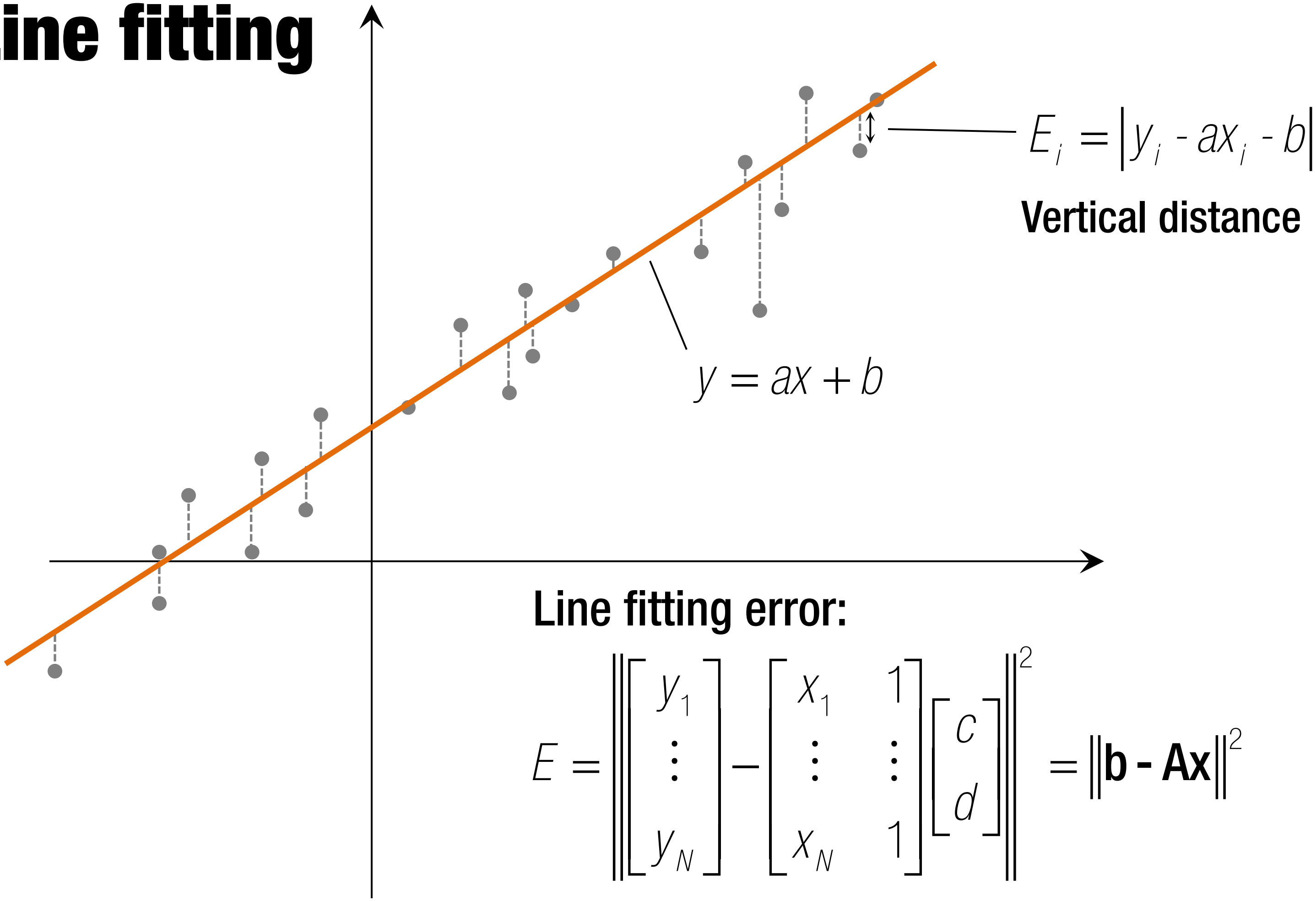
Line fitting



Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{Ax} \right\|^2$$

Line fitting

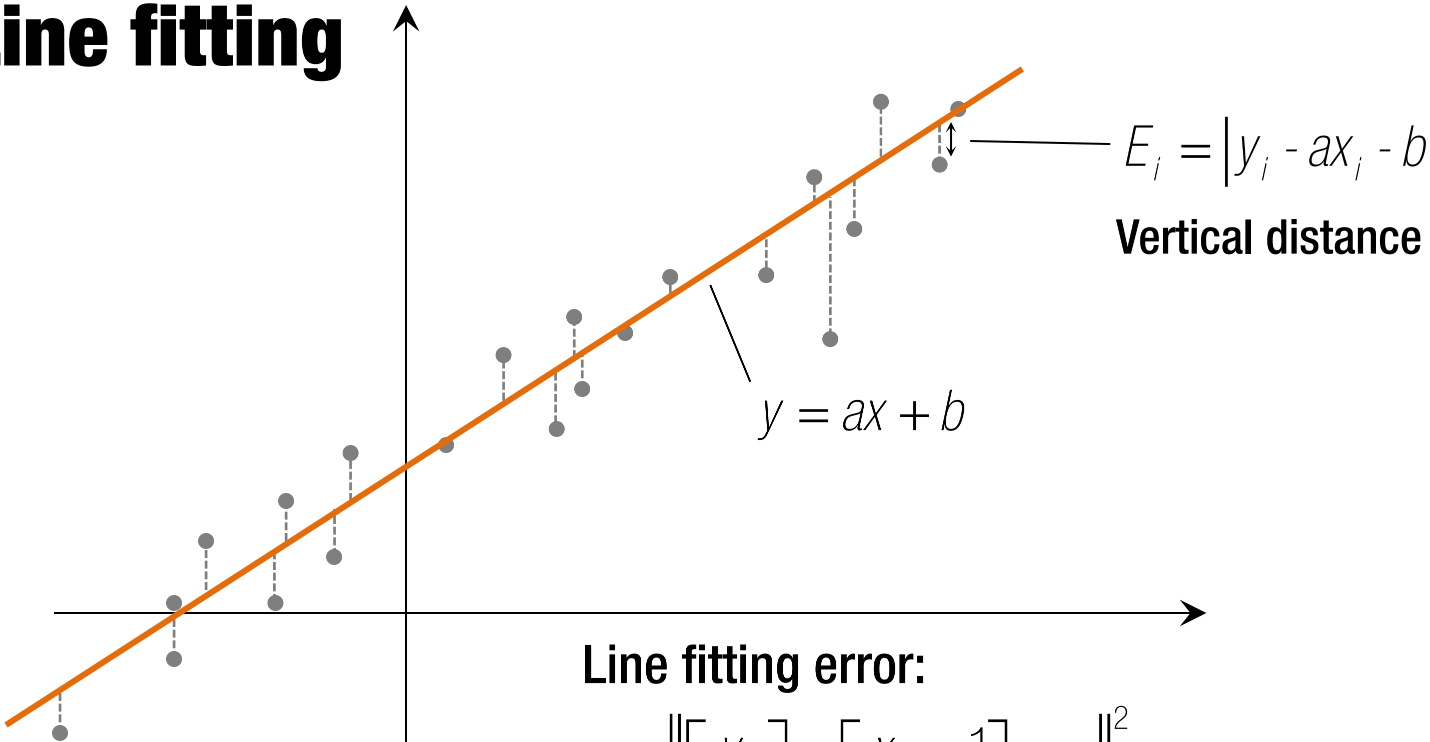


Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

Line fitting



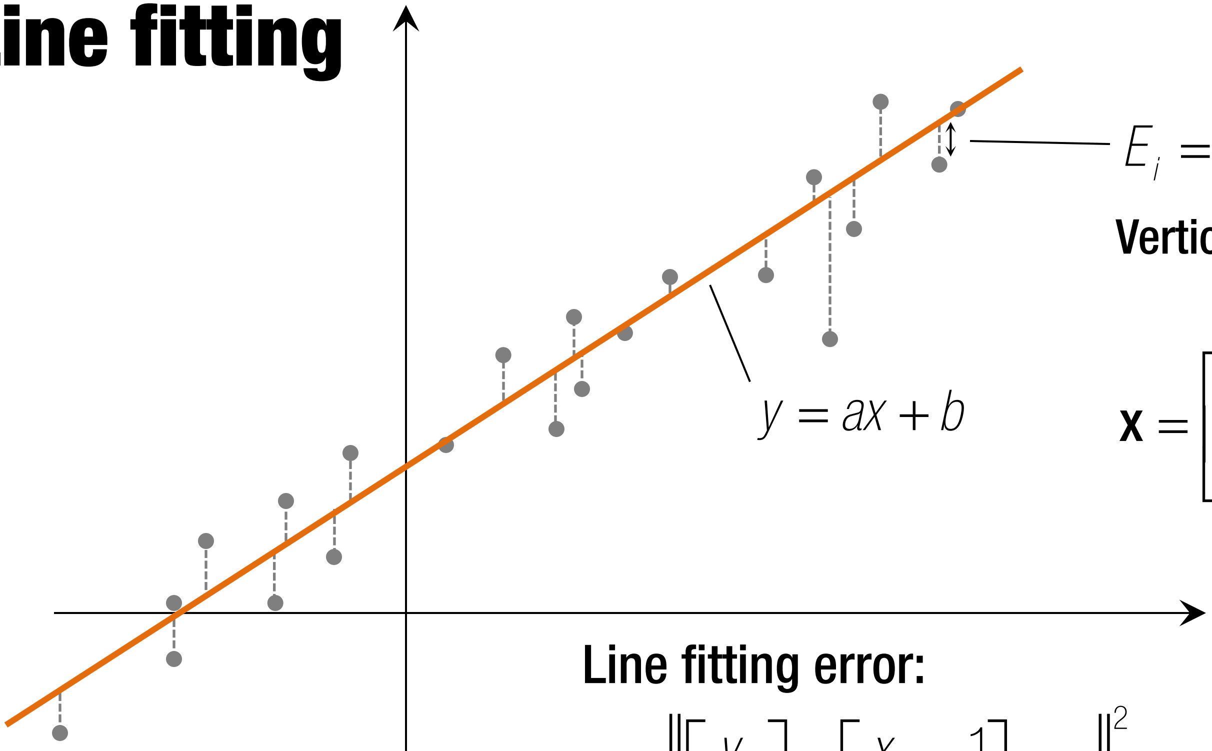
Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \|\mathbf{b} - \mathbf{Ax}\|^2$$

$$= (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{b}^\top \mathbf{b} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax}$$

$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^\top \mathbf{A} + 2\mathbf{A}^\top \mathbf{Ax} = \mathbf{0} \;\rightarrow\; \mathbf{A}^\top \mathbf{Ax} = \mathbf{b}^\top \mathbf{A}$$

Line fitting



$$\mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix} = \left(\mathbf{A}^{\top}\mathbf{A}\right)^{-1}\mathbf{b}^{\top}\mathbf{A}$$

Line fitting error:

$$E = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right\|^2 = \left\| \mathbf{b} - \mathbf{A}\mathbf{x} \right\|^2$$

$$= \left(\mathbf{b} - \mathbf{A}\mathbf{x}\right)^{\top}\left(\mathbf{b} - \mathbf{A}\mathbf{x}\right) = \mathbf{b}^{\top}\mathbf{b} - 2\mathbf{b}^{\top}\mathbf{A}\mathbf{x} + \mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{A}\mathbf{x}$$

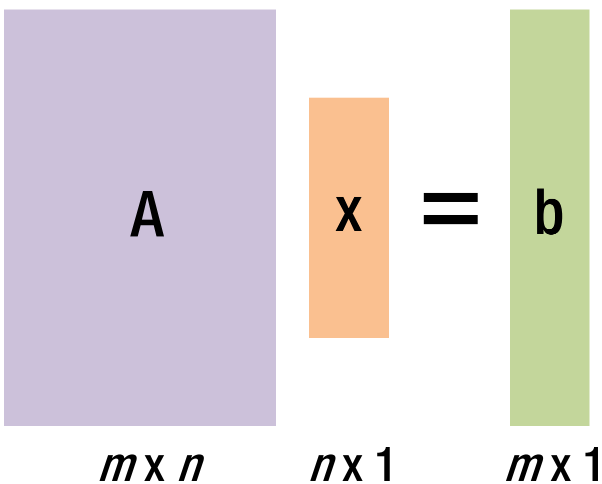
$$\frac{\partial E}{\partial \mathbf{x}} = -2\mathbf{b}^{\top}\mathbf{A} + 2\mathbf{A}^{\top}\mathbf{A}\mathbf{x} = \mathbf{0} \;\rightarrow\; \mathbf{A}^{\top}\mathbf{A}\mathbf{x} = \mathbf{b}^{\top}\mathbf{A}$$

Linear Inhomogeneous Equations

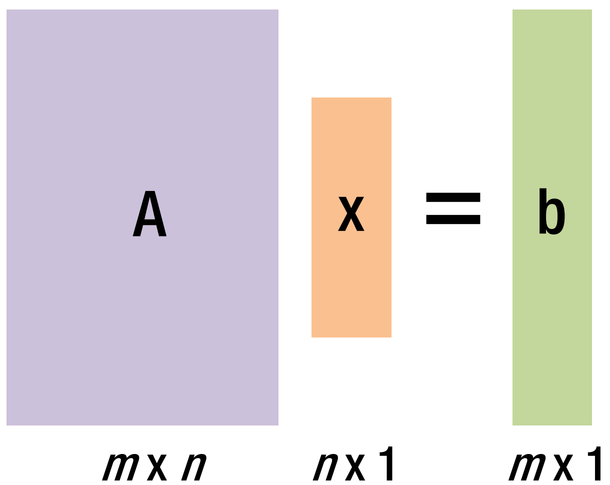
1) $\text{rank}(\mathbf{A}) = r < n$: infinite number of solutions

$$\mathbf{x} = \underbrace{\mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{\top}\mathbf{b}}_{\text{Particular solution}} + \underbrace{\lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n}_{\text{Homogeneous solution}}$$

where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}$ and $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_n \end{bmatrix}$.



Linear Inhomogeneous Equations



1) $\mathrm{rank}(\mathbf{A}) = r < n$: infinite number of solutions

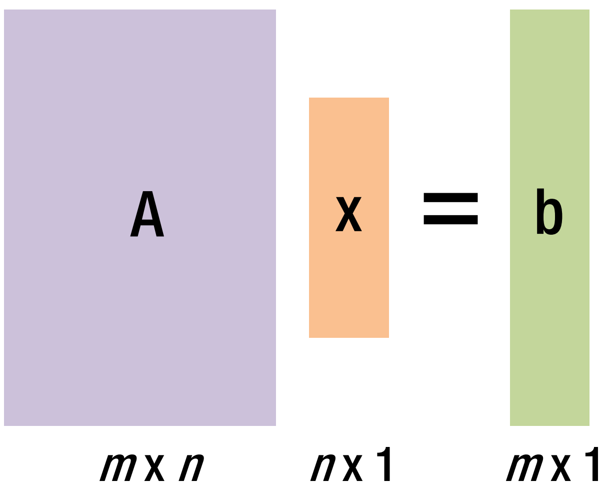
$$\mathbf{x} = \underbrace{\mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{\top}\mathbf{b}}_{\text{Particular solution}} + \underbrace{\lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_{n}\mathbf{V}_{n}}_{\text{Homogeneous solution}}$$

where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}$ and $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_n \end{bmatrix}$.

2) $\mathrm{rank}(\mathbf{A}) = n$: exact solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Linear Inhomogeneous Equations



1) $\mathrm{rank}(A) = r < n$: infinite number of solutions

$$x = \underbrace{VD^{-1}U^{T}b}_{\text{Particular solution}} + \underbrace{\lambda_{r+1}V_{r+1} + \cdots + \lambda_n V_n}_{\text{Homogeneous solution}}$$

where $A = UDV^{T}$ and $V = \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix}$.

2) $\mathrm{rank}(A) = n$: exact solution

$$x = A^{-1}b$$

3) $n < m$: no exact solution in general (needs least squa

$$\min_x \|Ax - b\|^2 \rightarrow x = \left(A^{T}A\right)^{-1} A^{T}b$$

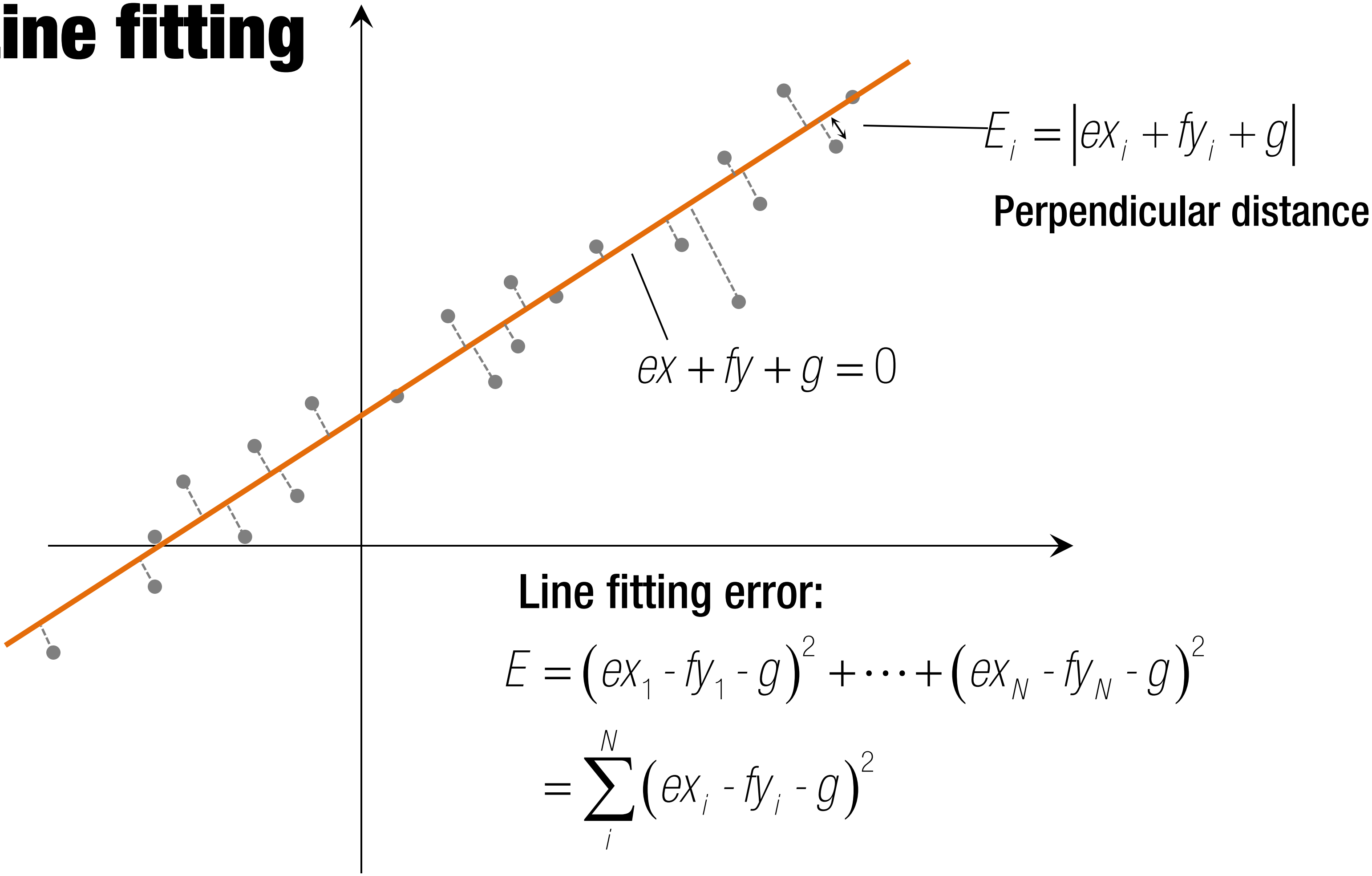
or $x = A \backslash b$ in MATLAB.

Two types of Least Square Problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathsf{A}\mathbf{x} - \mathbf{b}\|^2 \quad \text{with} \quad \|\mathbf{b}\| \neq 0$$

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathsf{A}\mathbf{x}\|^2 \quad \text{s.t.} \quad \|\mathbf{x}\| = 1$$

Line fitting

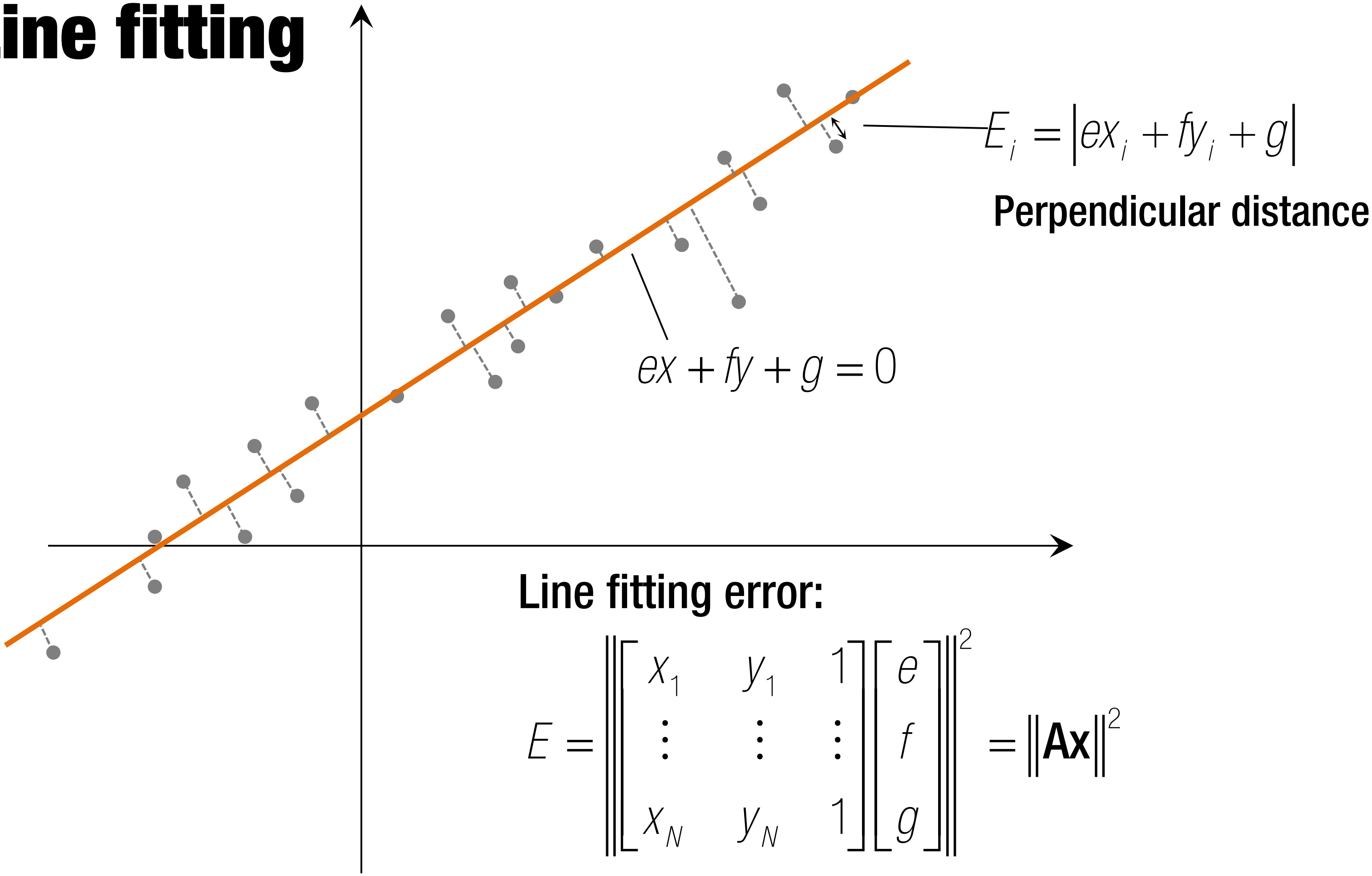


Line fitting error:

$$E = (ex_1 - fy_1 - g)^2 + \cdots + (ex_N - fy_N - g)^2$$

$$= \sum_{i}^{N} (ex_i - fy_i - g)^2$$

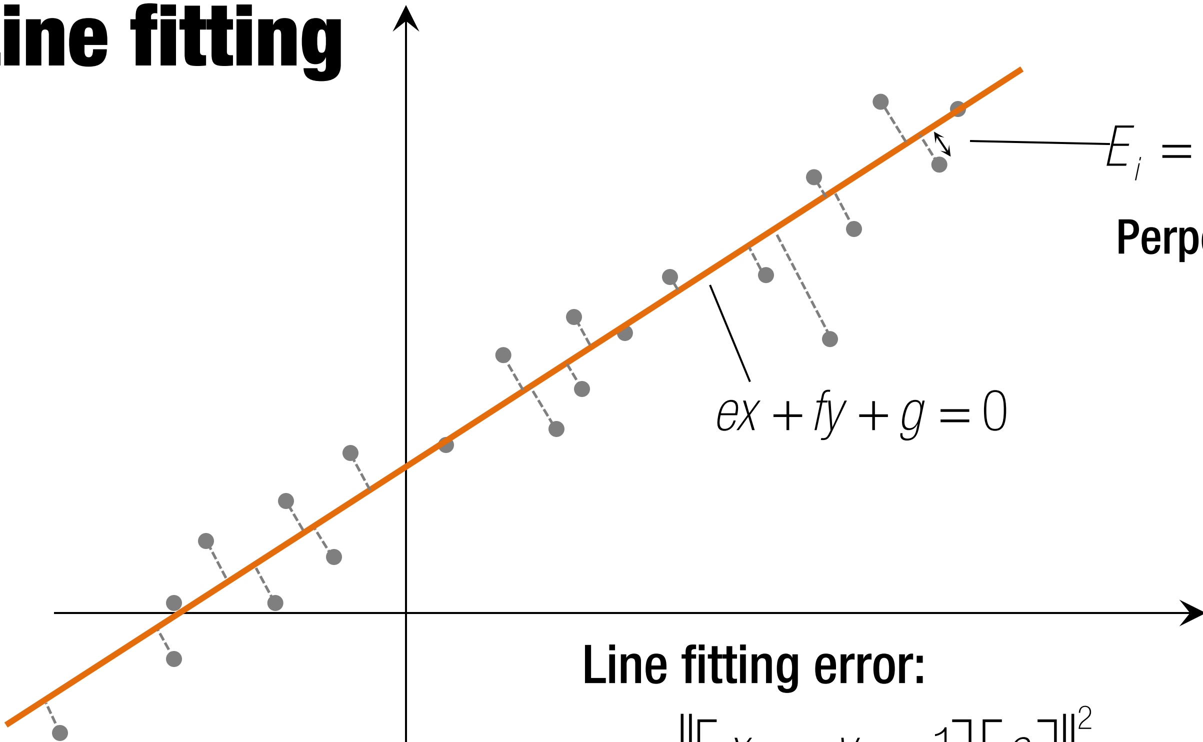
Line fitting



Line fitting error:

$$E = \left\| \begin{bmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\|^2 = \|\mathbf{Ax}\|^2$$

Line fitting



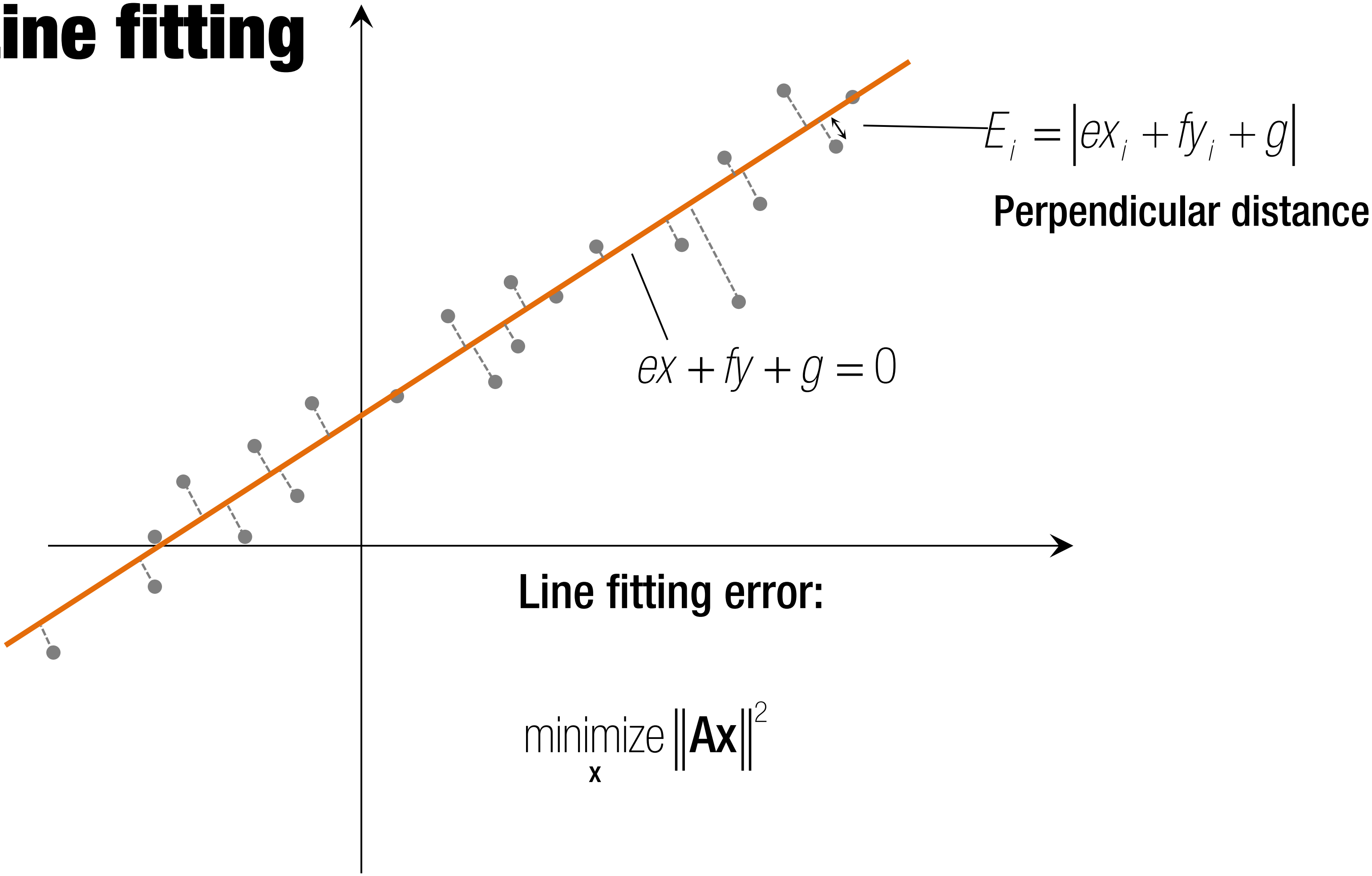
$$E_i = |ex_i + fy_i + g|$$

Perpendicular distance

Line fitting error:

$$E = \left\| \begin{bmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\|^2 = \|\mathbf{Ax}\|^2$$

Line fitting

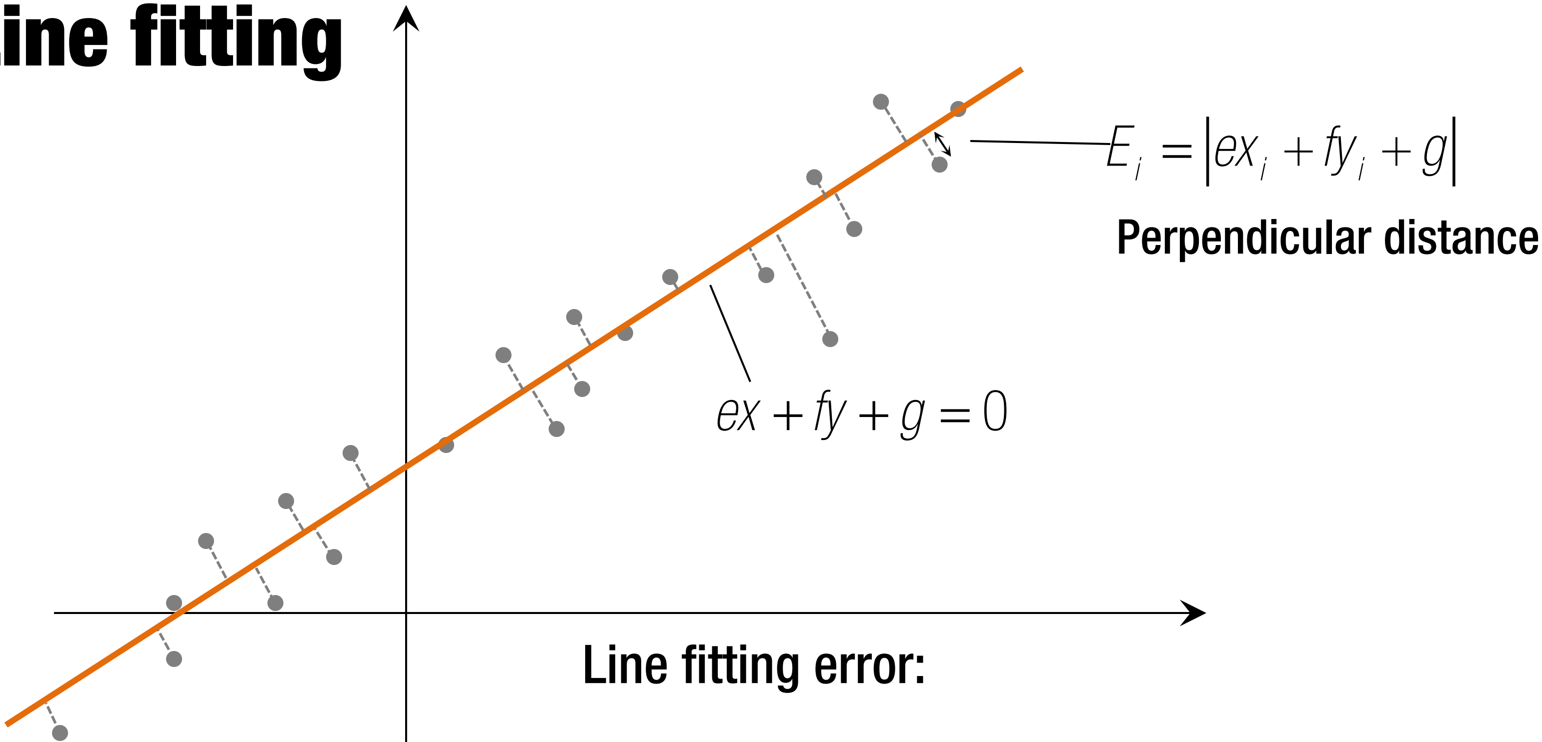


Line fitting error:

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

Line fitting



Line fitting error:

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

$$\text{subject to } \|\mathbf{x}\| = 1$$

$$\mathbf{x} = \mathbf{V}_3 \quad \text{where } \mathbf{A} = \mathbf{UDV}^{\mathsf{T}}$$

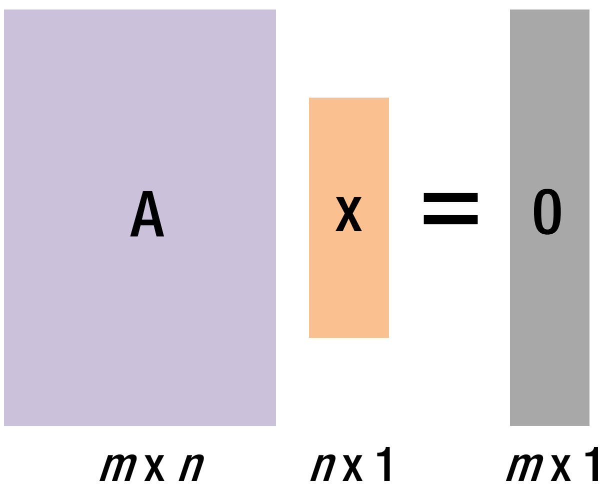
$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \end{bmatrix}$$

Linear Homogeneous Equations

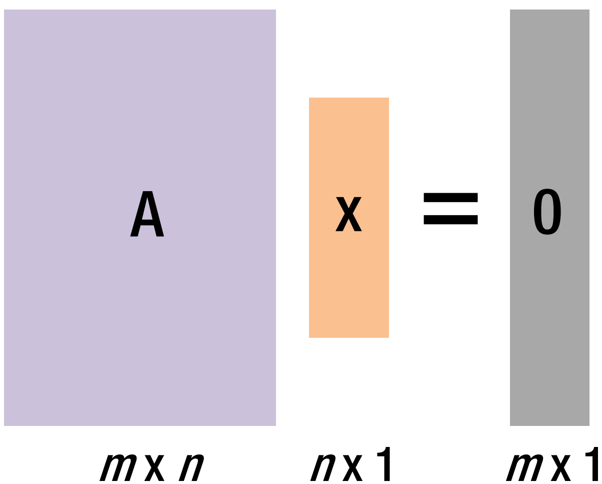
Linear least square solve produces a trivial solution:

$$\mathbf{x} = \left(\mathbf{A}^{\top}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on $\mathbf{x}$ to avoid the trivial solution: $\left\|\mathbf{x}\right\| = 1$



Linear Homogeneous Equations



Linear least square solve produces a trivial solution:

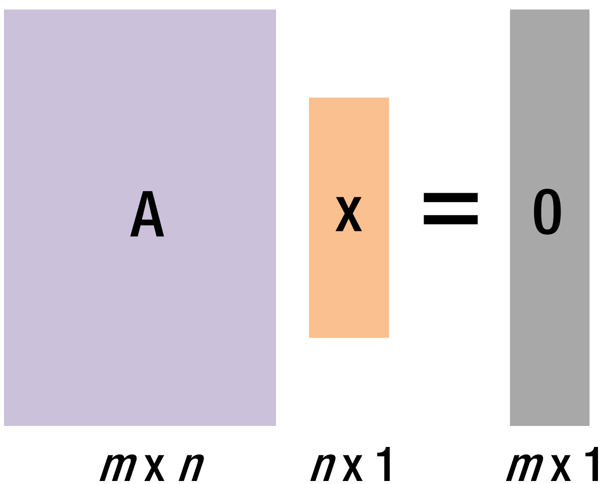
$$\mathbf{x} = \left(\mathbf{A}^{\top}\mathbf{A}\right)^{-1}\mathbf{A}^{\top}\mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on $\mathbf{x}$ to avoid the trivial solution: $\left\|\mathbf{x}\right\| = 1$

1) $\text{rank}(\mathbf{A}) = r < n-1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_{n}\mathbf{V}_{n} \quad \text{where} \quad \sum_{i=r+1}^{n} \lambda_i^2 = 1$$

Linear Homogeneous Equations



Linear least square solve produces a trivial solution:

$$\mathbf{x} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on $\mathbf{x}$ to avoid the trivial solution: $\|\mathbf{x}\| = 1$

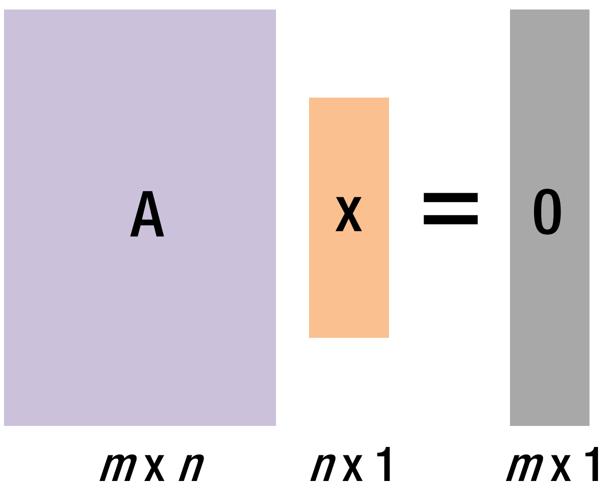
1) $\text{rank}(\mathbf{A}) = r < n-1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n \quad \text{where} \quad \sum_{i=r+1}^{n} \lambda_i^2 = 1$$

2) $\text{rank}(\mathbf{A}) = n-1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

Linear Homogeneous Equations



Linear least square solve produces a trivial solution:

$$\mathbf{x} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{b} \rightarrow \mathbf{x} = \mathbf{0}$$

An additional constraint on $\mathbf{x}$ to avoid the trivial solution: $\|\mathbf{x}\| = 1$

1) $\text{rank}(\mathbf{A}) = r < n-1$: infinite number of solutions

$$\mathbf{x} = \lambda_{r+1}\mathbf{V}_{r+1} + \cdots + \lambda_n \mathbf{V}_n \quad \text{where} \quad \sum_{i=r+1}^{n} \lambda_i^2 = 1$$

2) $\text{rank}(\mathbf{A}) = n-1$: one exact solution

$$\mathbf{x} = \mathbf{V}_n$$

3) $n < m$: no exact solution in general (needs least squar

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\| = 1 \rightarrow \mathbf{x} = \mathbf{V}_n$$