Lecture 3

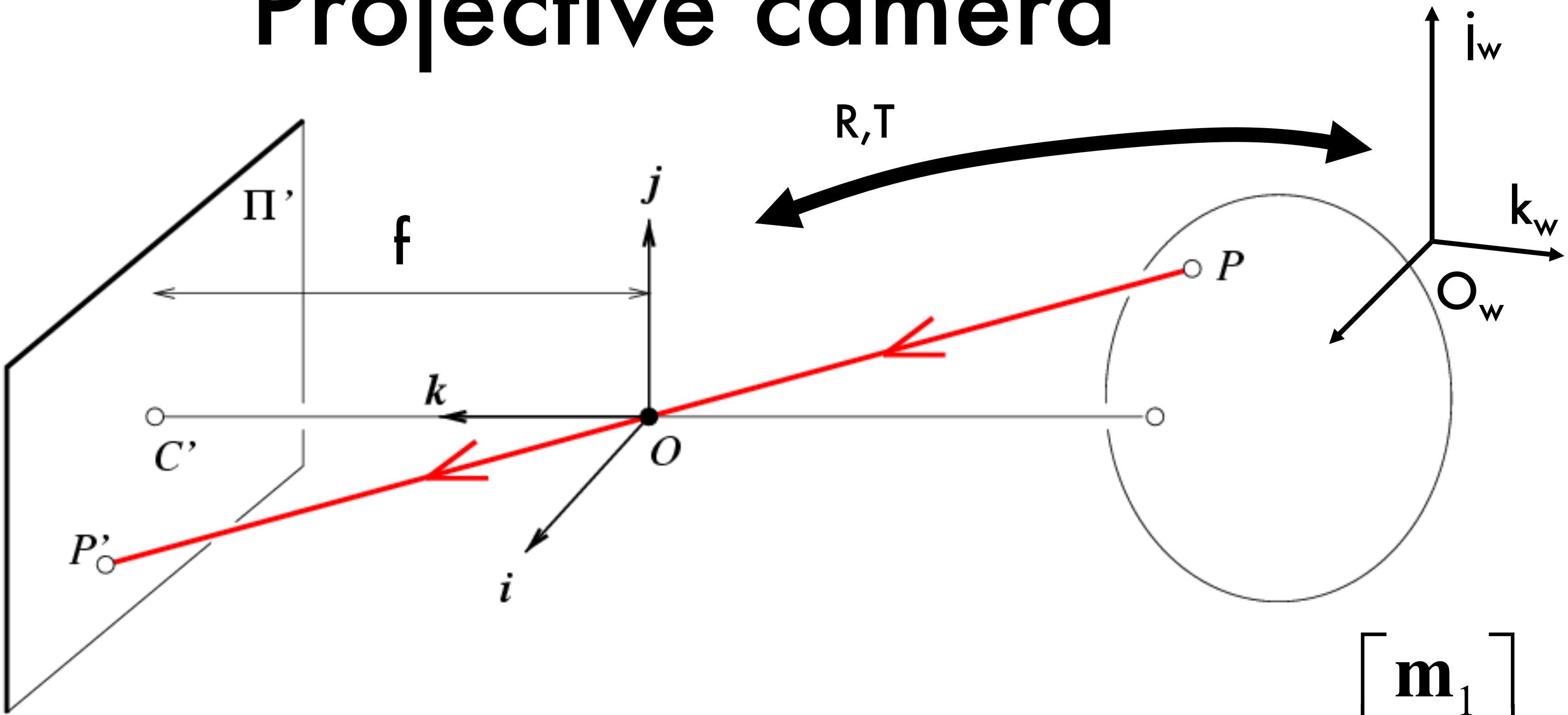
Camera Models 2 & Camera Calibration

- Recap of camera models
- Camera calibration problem
- Camera calibration with radial distortion
- Example

Reading: [FP] Chapter 1 "Geometric Camera Calibration"
[HZ] Chapter 7 "Computation of Camera Matrix P"

Some slides in this lecture are courtesy to Profs. J. Ponce, F-F Li

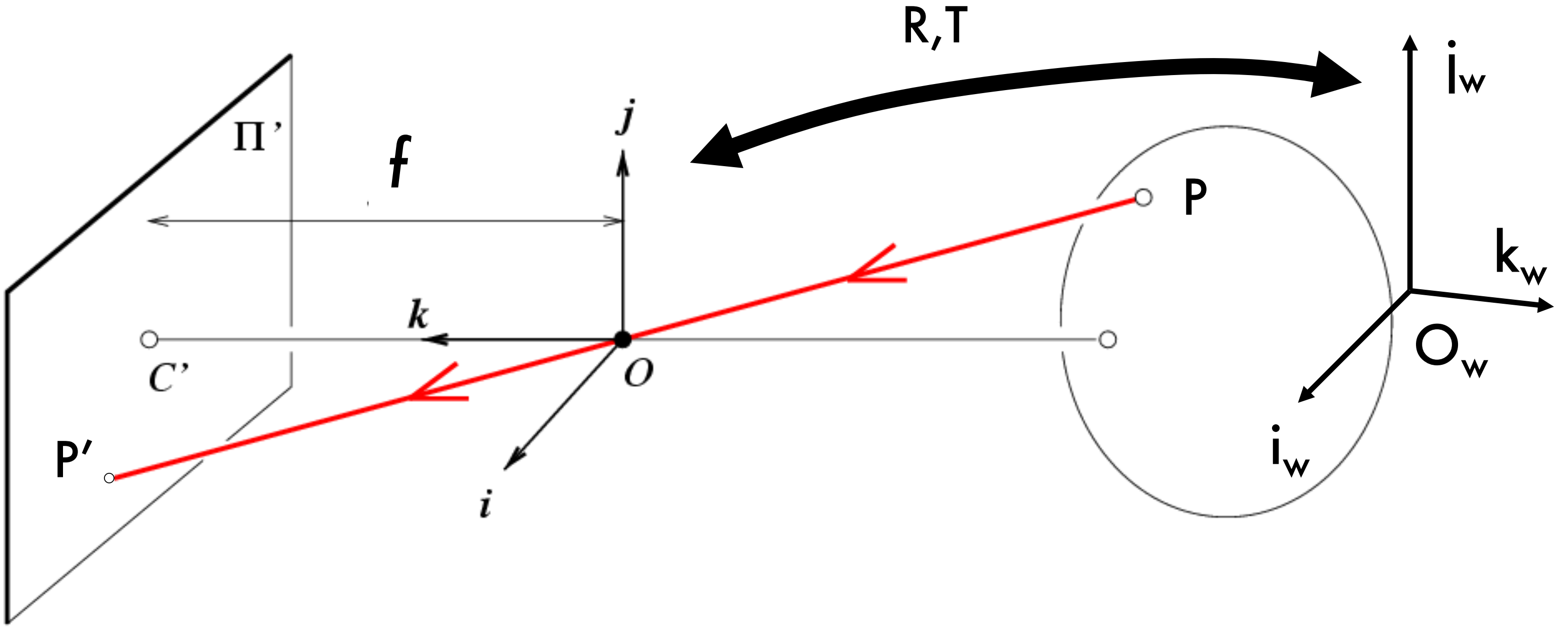
Projective camera



$$P'_{3\times1} = M\ P_w \quad = K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4} P_{w\,4\times1} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_\mathrm{W} = \begin{bmatrix} \mathbf{m}_1 P_\mathrm{W} \\ \mathbf{m}_2 P_\mathrm{W} \\ \mathbf{m}_3 P_\mathrm{W} \end{bmatrix} \xrightarrow{\mathbf{E}} P'_E = \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right)$$

Exercise!



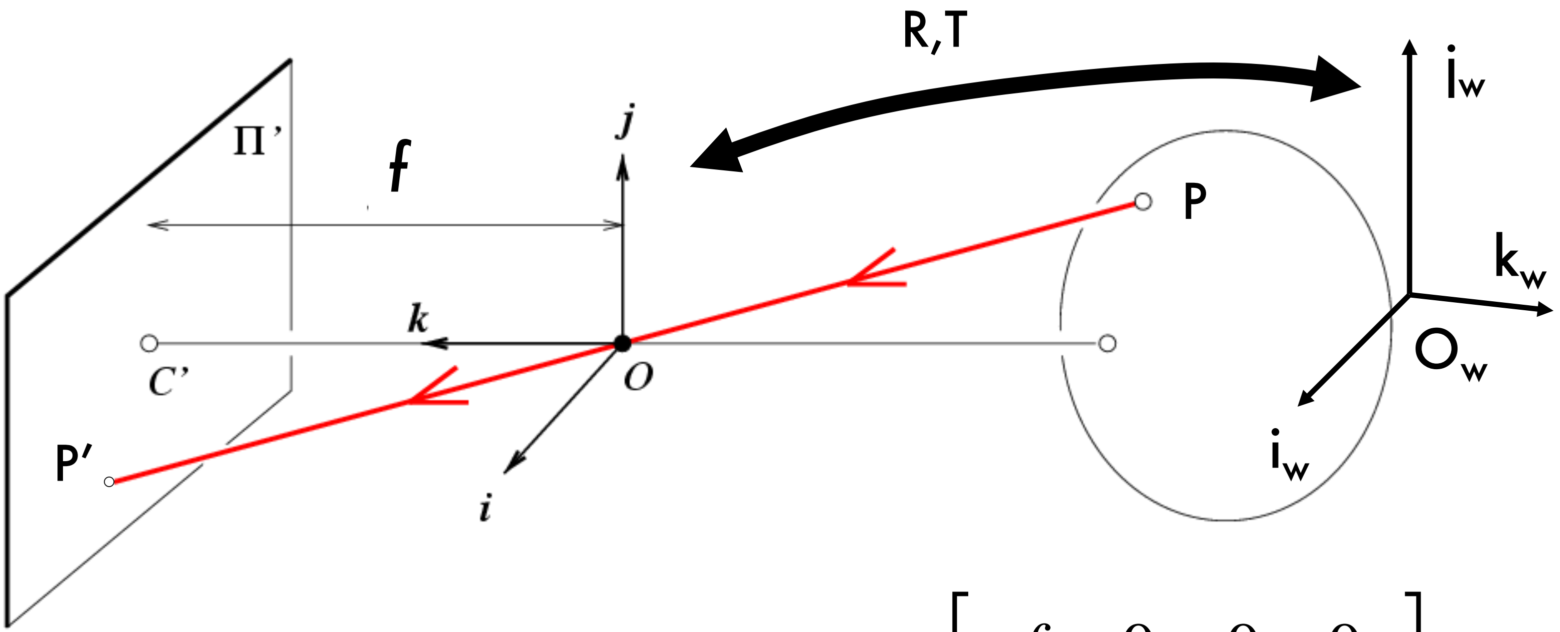
$$M = K\begin{bmatrix} R & T \end{bmatrix}$$

Suppose we have no rotation or translation
Zero skew, square pixels, no distortion, no off-set

$$P_w = \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

$$\rightarrow \; P'_E = \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right)$$

Exercise!



$$M = K\begin{bmatrix} R & T \end{bmatrix} = K\begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \; P'_E = \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right) = \left(f\frac{x_w}{z_w}, f\frac{y_w}{z_w}\right)$$

$$P_w = \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Canonical Projective Transformation

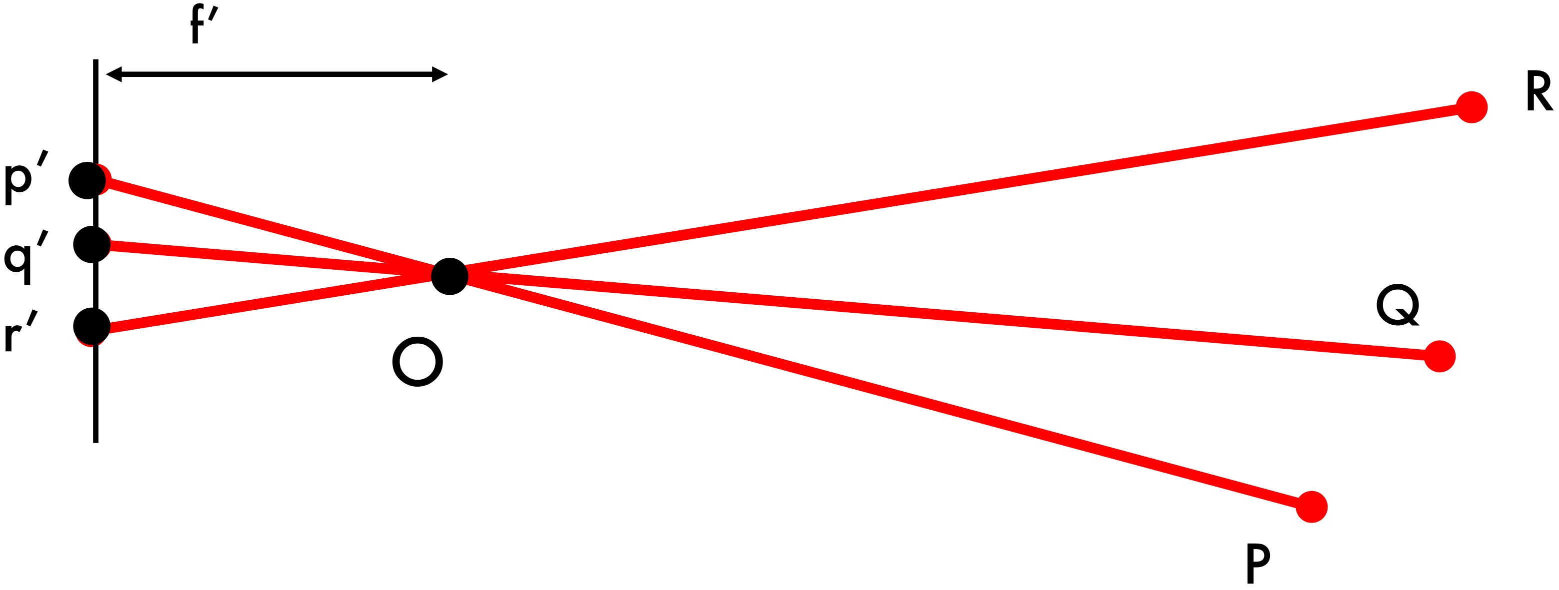
$$P' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = M\ P$$

$$\Re^4 \xrightarrow{\text{H}} \Re^3$$

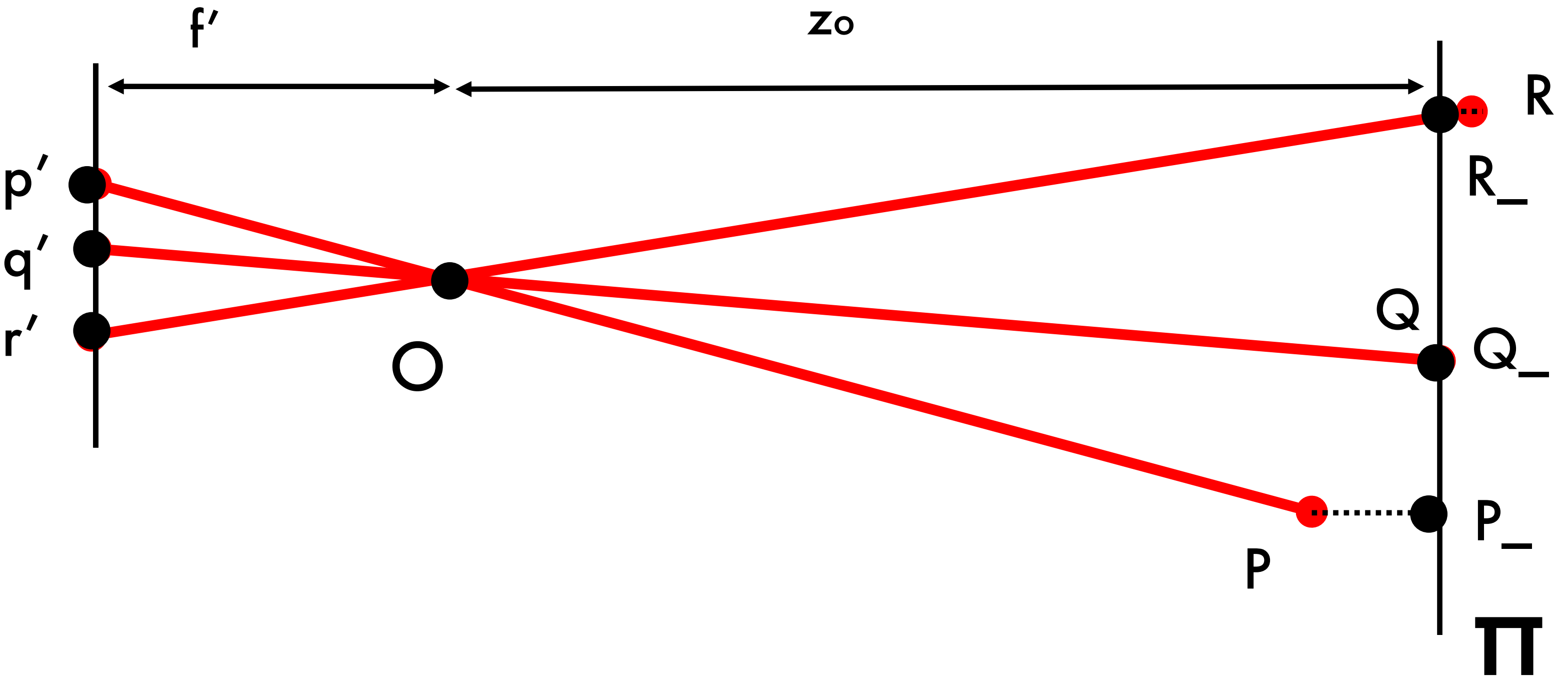
$$P_i' = \begin{bmatrix} \frac{x}{z} \\ \frac{y}{z} \end{bmatrix}$$

Projective camera

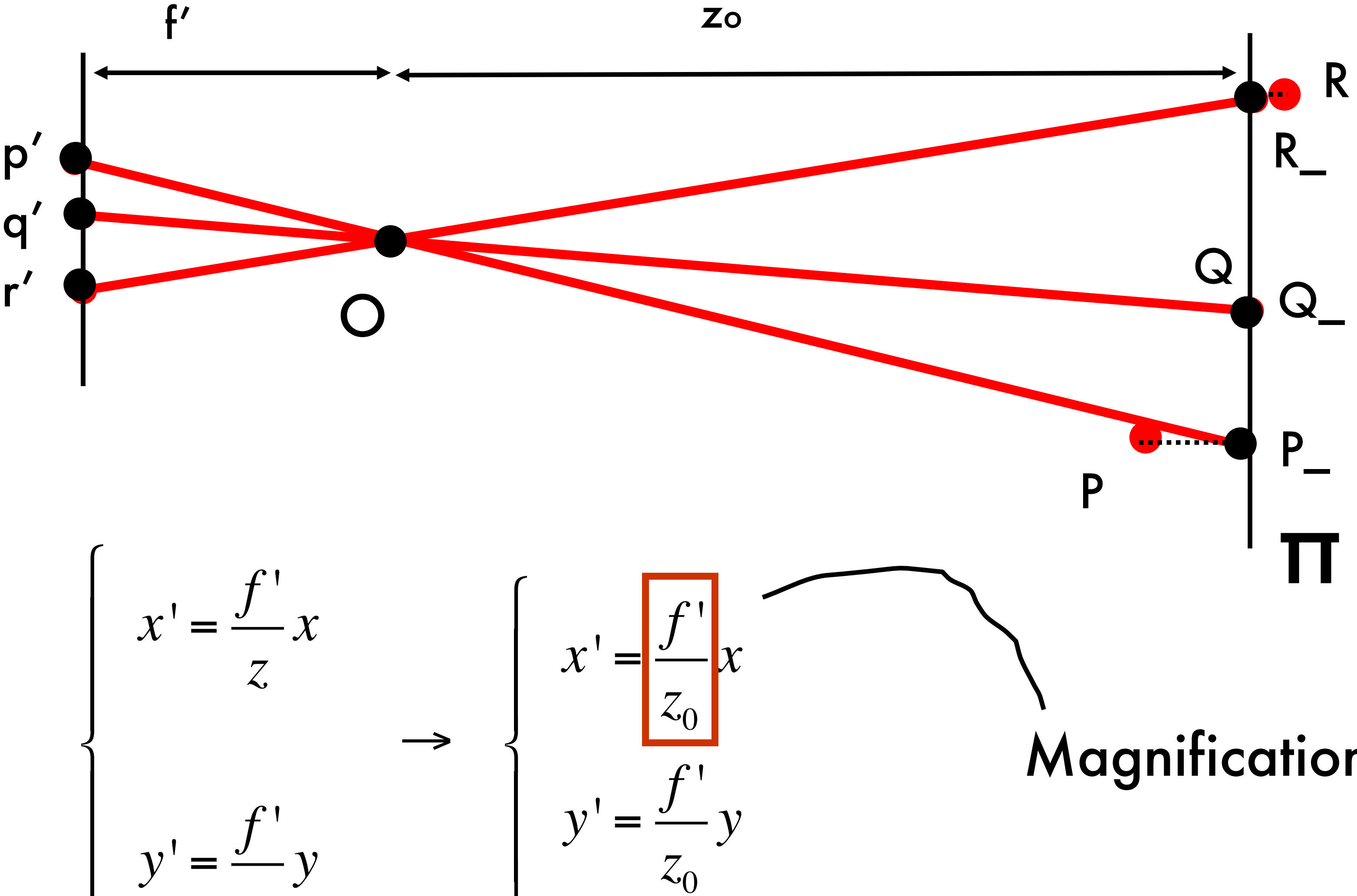


Weak perspective projection

When the relative scene depth is small compared to its distance from the camera



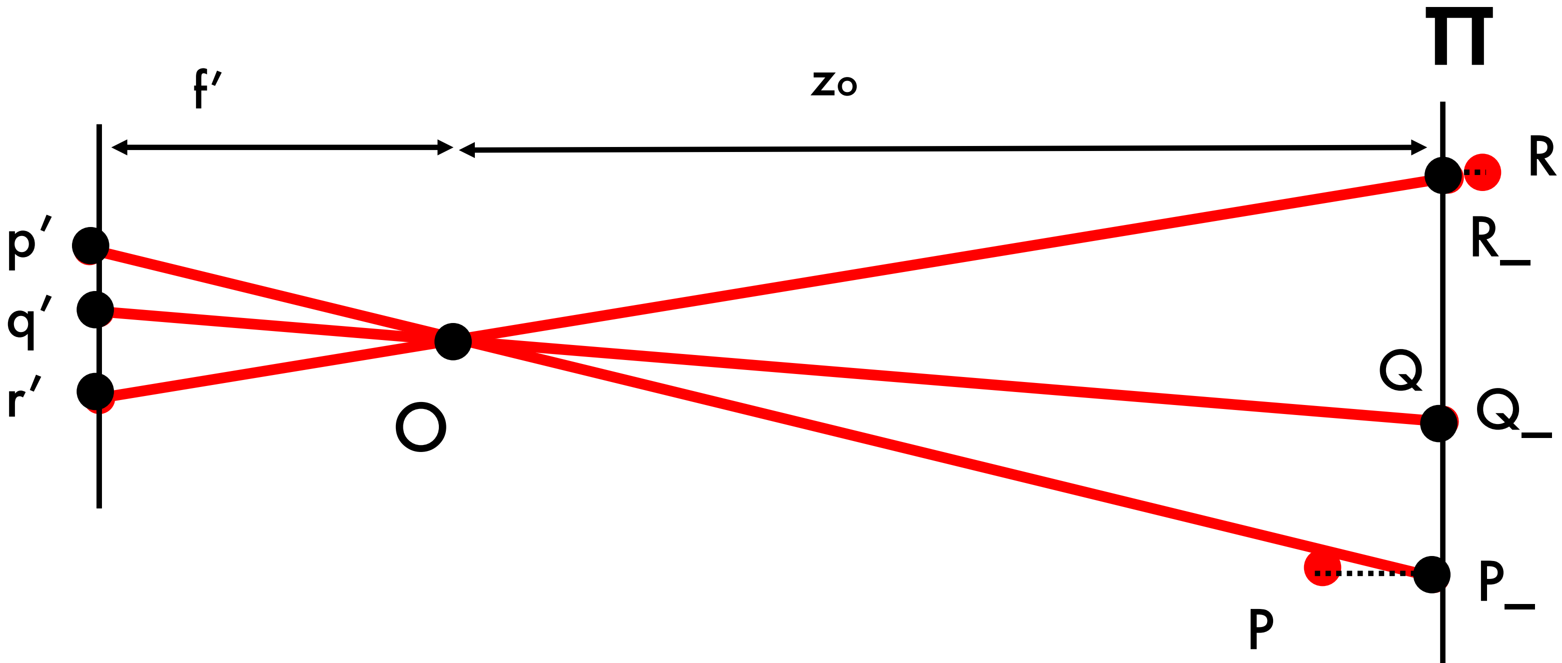
Weak perspective projection



$$\begin{cases} x' = \dfrac{f'}{z}x \\ y' = \dfrac{f'}{z}y \end{cases} \rightarrow \begin{cases} x' = \boxed{\dfrac{f'}{z_0}}x \\ y' = \dfrac{f'}{z_0}y \end{cases}$$

Magnification m

Weak perspective projection



Projective (perspective)

Weak perspective

$$M = K\begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{v} & \mathbf{1} \end{bmatrix} \rightarrow \quad M = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$P' = M\,P_w = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_W = \begin{bmatrix} \mathbf{m}_1\,P_W \\ \mathbf{m}_2\,P_W \\ \mathbf{m}_3\,P_W \end{bmatrix} \qquad M = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{v} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

$$\xrightarrow{\mathbf{E}} \left(\frac{\mathbf{m}_1\,P_w}{\mathbf{m}_3\,P_w}, \frac{\mathbf{m}_2\,P_w}{\mathbf{m}_3\,P_w}\right)$$

Perspective

$$P' = M\,P_w = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_W = \begin{bmatrix} \mathbf{m}_1\,P_W \\ \mathbf{m}_2\,P_W \\ 1 \end{bmatrix} \qquad M = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} = \begin{bmatrix} & & \mathbf{m}_1 & \\ & & \mathbf{m}_2 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

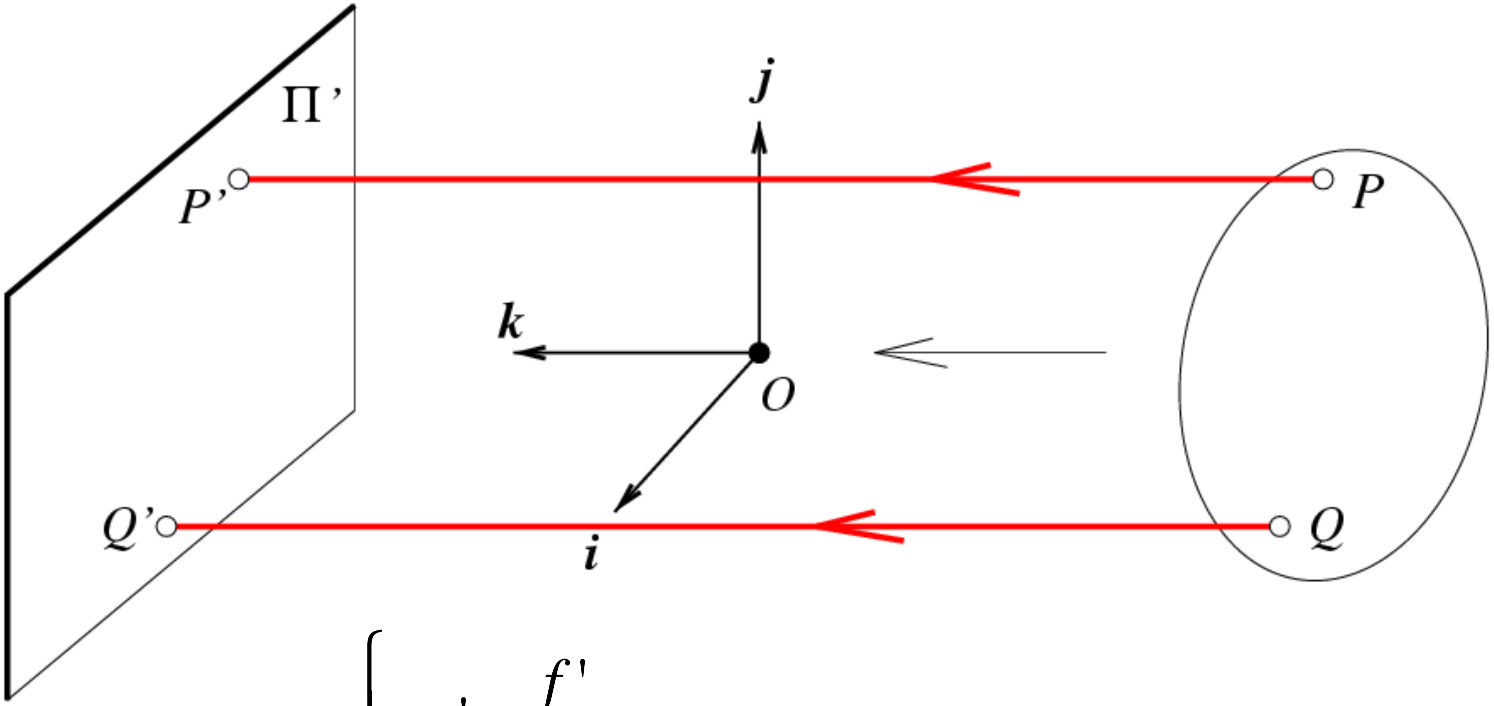
$$\xrightarrow{\mathbf{E}} (\underset{\uparrow}{\mathbf{m}_1}\,P_w, \underset{\uparrow}{\mathbf{m}_2}\,P_w)$$

magnification

Weak perspective

Orthographic (affine) projection

Distance from center of projection to image plane is infinite



$$\begin{cases} x' = \dfrac{f'}{z} x \\ \\ y' = \dfrac{f'}{z} y \end{cases} \rightarrow \begin{cases} x' = x \\ y' = y \end{cases}$$

Pros and Cons of These Models

- Weak perspective results in much simpler math.
  - Accurate when object is small and distant.
  - Most useful for recognition.

- Pinhole perspective is much more accurate for modeling the 3D-to-2D mapping.
  - Used in structure from motion or SLAM.

Lecture 3

Camera Calibration

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Why is this important?

Estimate camera parameters such pose or focal length from images!

Projective camera

$$P' = M\,P_w = K\begin{bmatrix} R & T \end{bmatrix} P_w$$

Internal parameters

External parameters

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}_{3\times 4}$$

$$K = \begin{bmatrix} \alpha & -\alpha\cot\theta & u_o \\ 0 & \dfrac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Goal of calibration

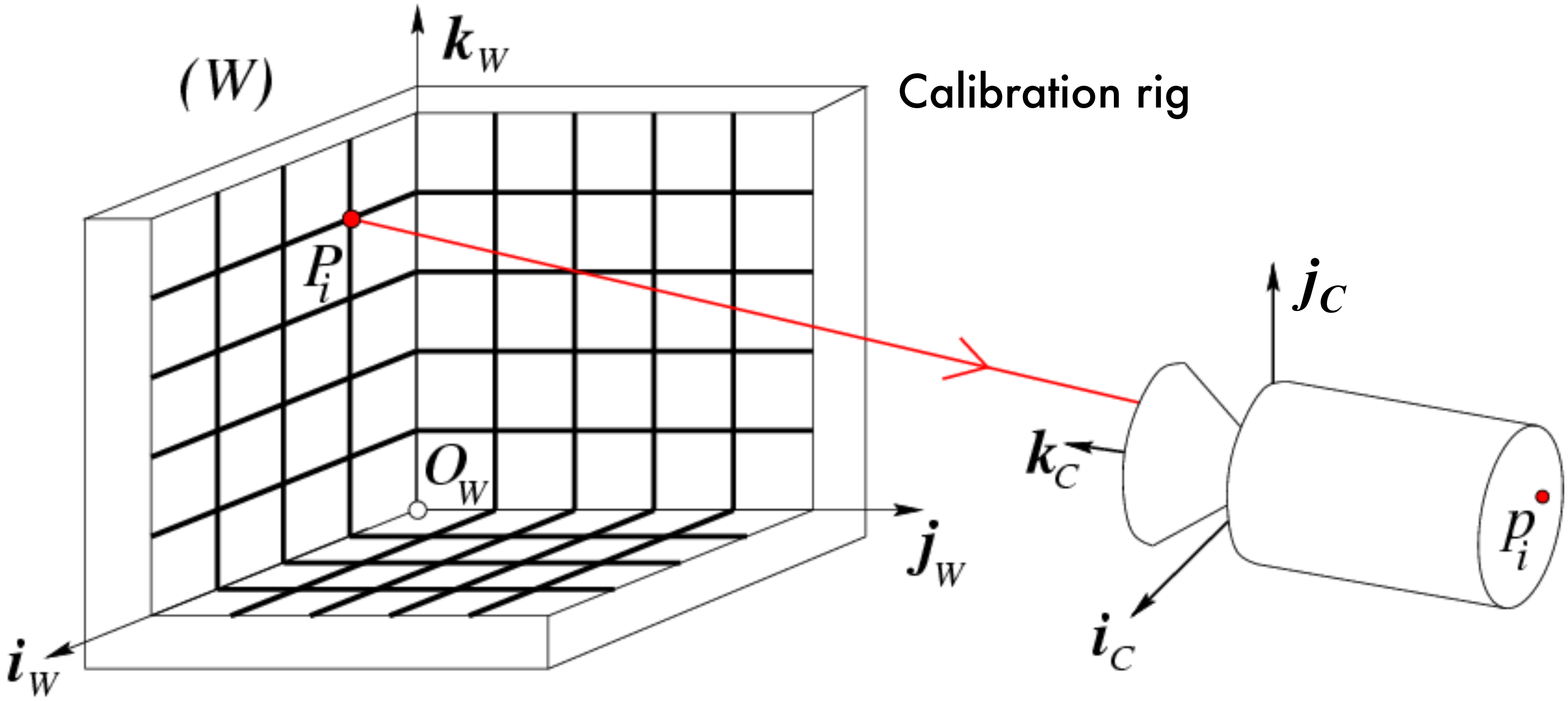
$$P' = M\,P_w \quad = \underbrace{K}_{\text{Internal parameters}}\underbrace{\begin{bmatrix} R & T \end{bmatrix}}_{\text{External parameters}} P_w$$

Estimate intrinsic and extrinsic parameters from 1 or multiple images

Change notation:
$P = P_w$

$p = P'$

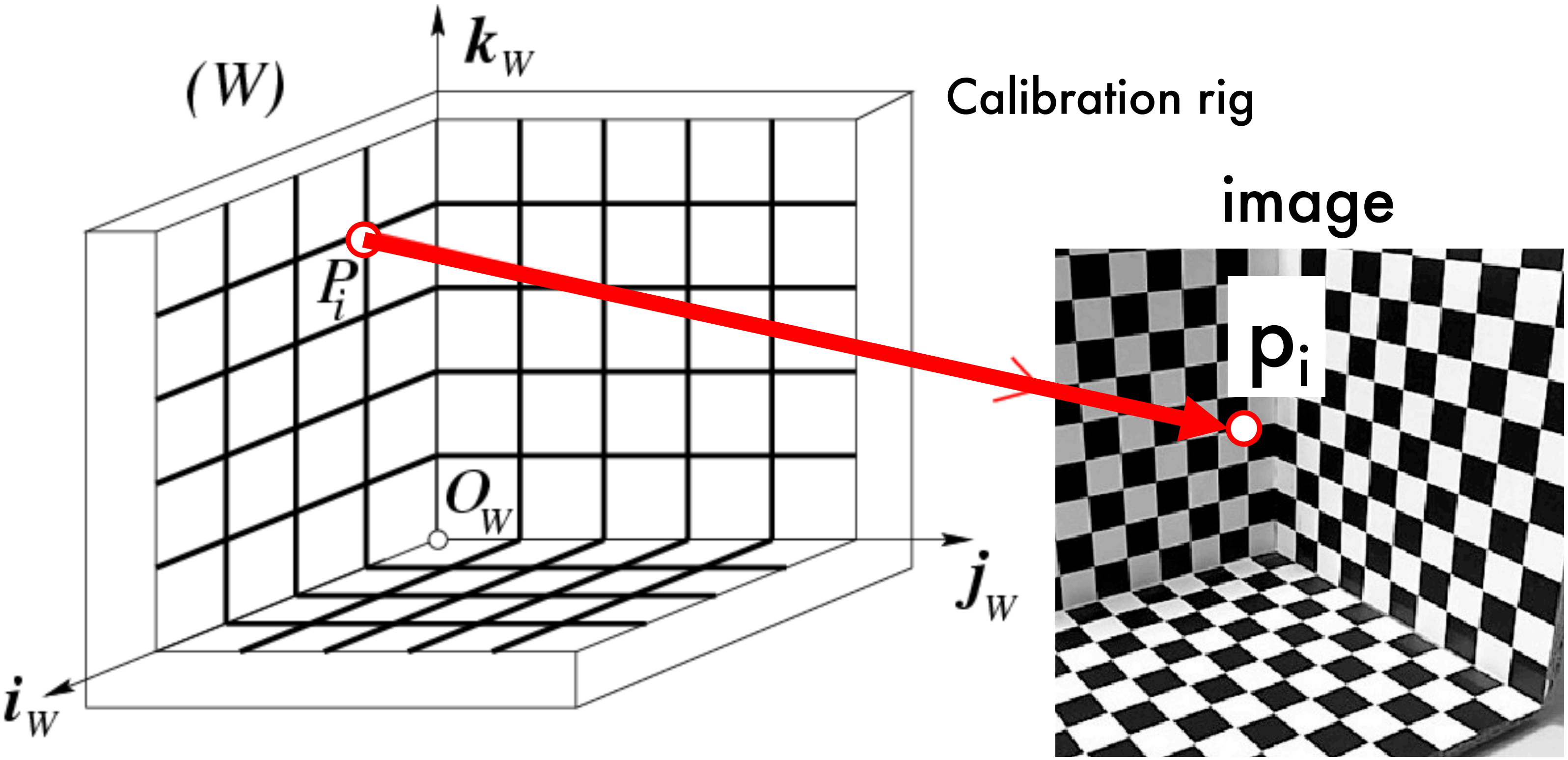
Calibration Problem



Calibration rig

- P_1… P_n with known positions in $[O_W, i_W, j_W, k_W]$

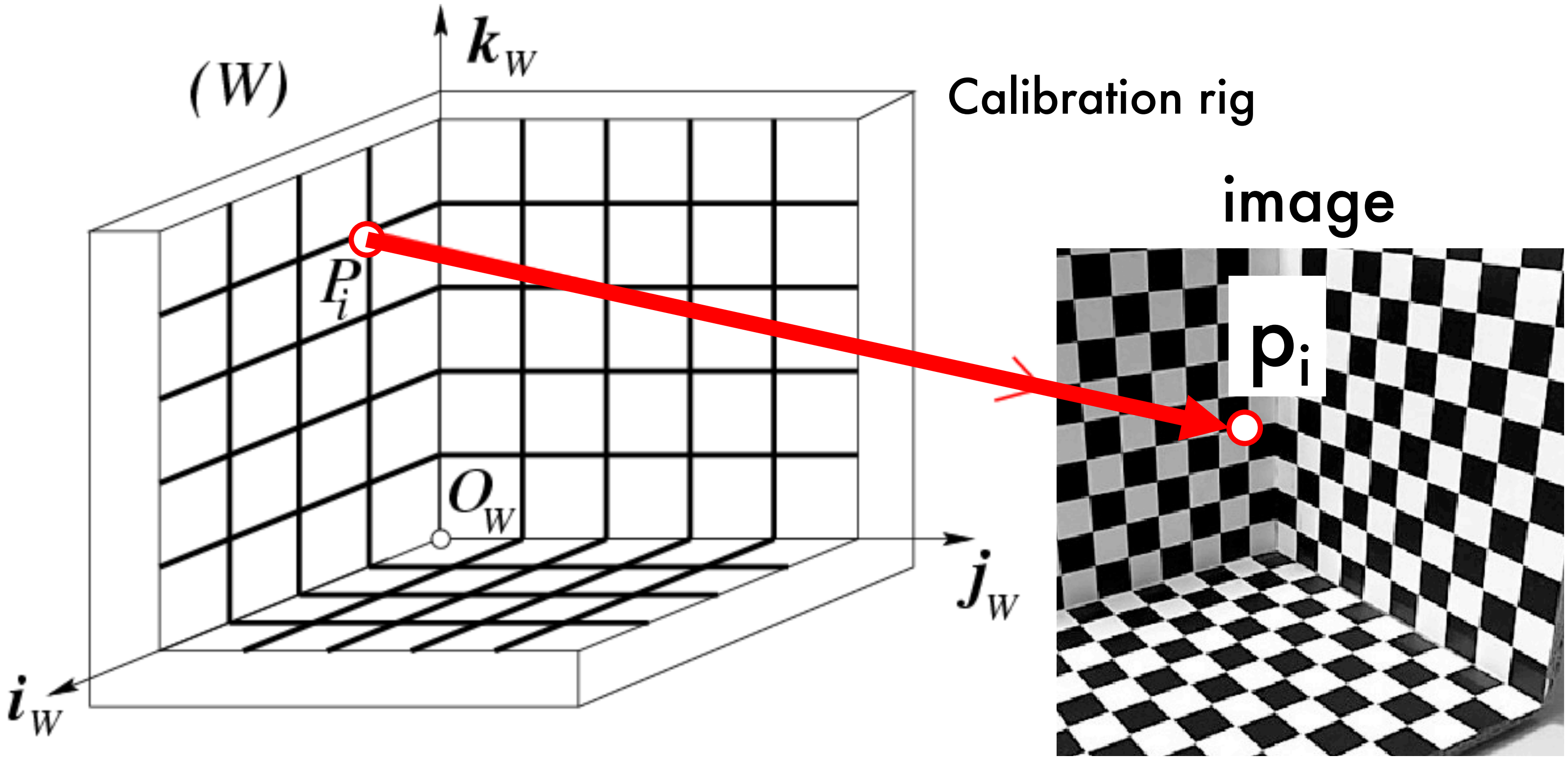
Calibration Problem



- P_1… P_n with known positions in $[O_w,i_w,j_w,k_w]$
- p_1, … p_n known positions in the image

Goal: compute intrinsic and extrinsic parameters

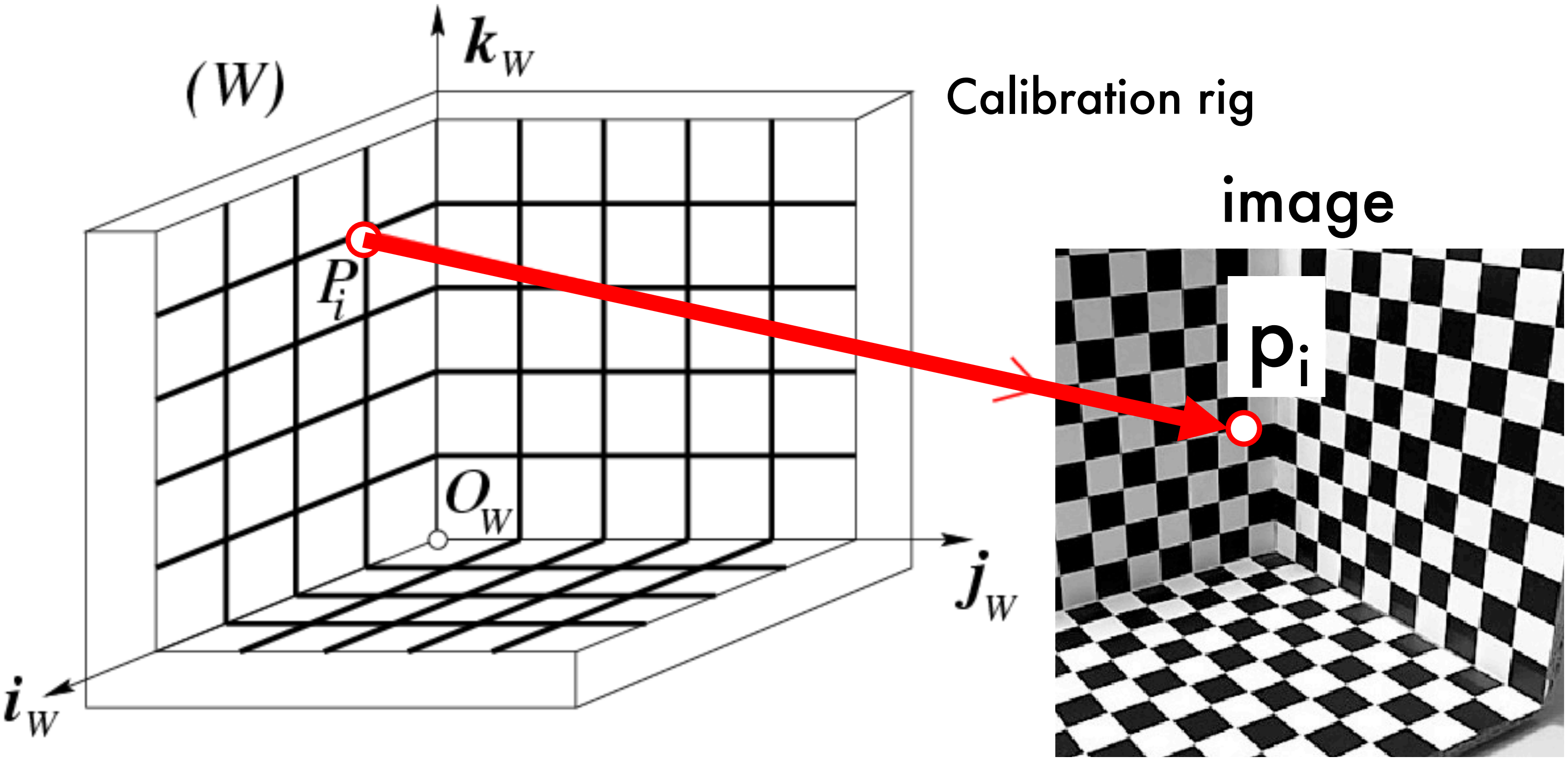
Calibration Problem



How many correspondences do we need?

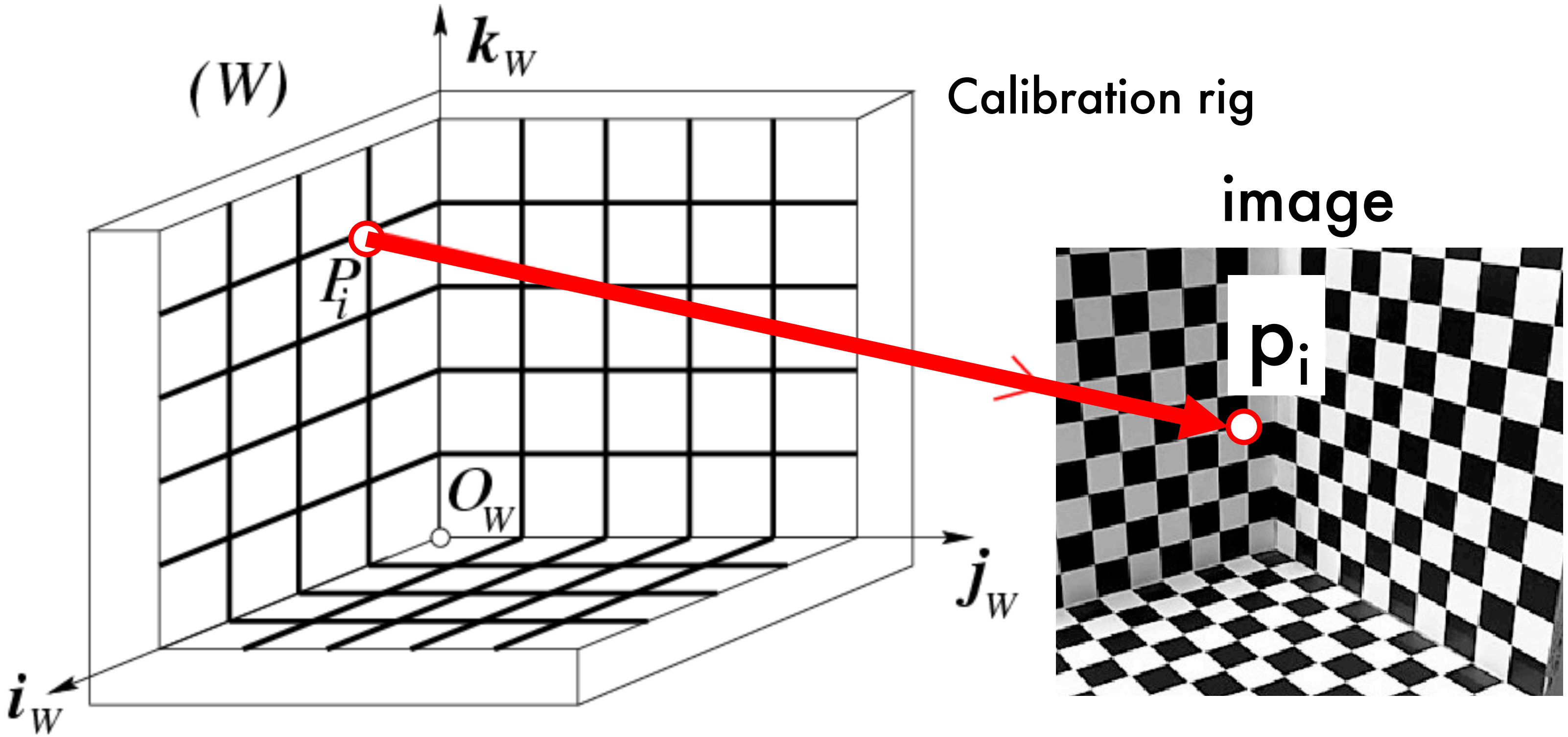
- M has 11 unknowns
- We need 11 equations
- 6 correspondences would do it

Calibration Problem



In practice, using more than 6 correspondences enables more robust results

Calibration Problem



$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{m}_1 \, \mathrm{P_i}}{\mathbf{m}_3 \, \mathrm{P_i}} \\ \dfrac{\mathbf{m}_2 \, \mathrm{P_i}}{\mathbf{m}_3 \, \mathrm{P_i}} \end{bmatrix} = M \; P_i$$

in pixels

[Eq. 1]

$$\mathrm{M} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

Calibration Problem

[Eq. 1]

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{m}_1 \, P_i}{\mathbf{m}_3 \, P_i} \\ \dfrac{\mathbf{m}_2 \, P_i}{\mathbf{m}_3 \, P_i} \end{bmatrix}$$

$$u_i = \frac{\mathbf{m}_1 \, P_i}{\mathbf{m}_3 \, P_i} \rightarrow u_i(\mathbf{m}_3 \, P_i) = \mathbf{m}_1 \, P_i \rightarrow u_i(\mathbf{m}_3 \, P_i) - \mathbf{m}_1 \, P_i = 0$$

$$v_i = \frac{\mathbf{m}_2 \, P_i}{\mathbf{m}_3 \, P_i} \rightarrow v_i(\mathbf{m}_3 \, P_i) = \mathbf{m}_2 \, P_i \rightarrow v_i(\mathbf{m}_3 \, P_i) - \mathbf{m}_2 \, P_i = 0$$

[Eqs. 2]

Calibration Problem

$$\begin{cases} u_1(\mathbf{m}_3\ P_1) - \mathbf{m}_1\ P_1 = 0 \\ v_1(\mathbf{m}_3\ P_1) - \mathbf{m}_2\ P_1 \ = 0 \\ \vdots \\ u_i(\mathbf{m}_3\ P_i) - \mathbf{m}_1\ P_i = 0 \\ v_i(\mathbf{m}_3\ P_i) - \mathbf{m}_2\ P_i \ = 0 \\ \vdots \\ u_n(\mathbf{m}_3\ P_n) - \mathbf{m}_1\ P_n = 0 \\ v_n(\mathbf{m}_3\ P_n) - \mathbf{m}_2\ P_n \ = 0 \end{cases}$$

[Eqs. 3]

Block Matrix Multiplication

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{bmatrix}$$

What is *AB* ?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Calibration Problem

$$\begin{cases} -u_1(\mathbf{m}_3\ P_1)+\mathbf{m}_1\ P_1=0 \\ -v_1(\mathbf{m}_3\ P_1)+\mathbf{m}_2\ P_1\ \ =0 \\ \qquad\vdots \\ -u_n(\mathbf{m}_3\ P_n)+\mathbf{m}_1\ P_n=0 \\ -v_n(\mathbf{m}_3\ P_n)+\mathbf{m}_2\ P_n\ \ =0 \end{cases} \longrightarrow \boxed{\overset{\text{known}}{\boxed{\mathbf{P}}}\ \overset{\text{unknown}}{\boxed{\mathbf{m}}}=0}$$

[Eq. 4]

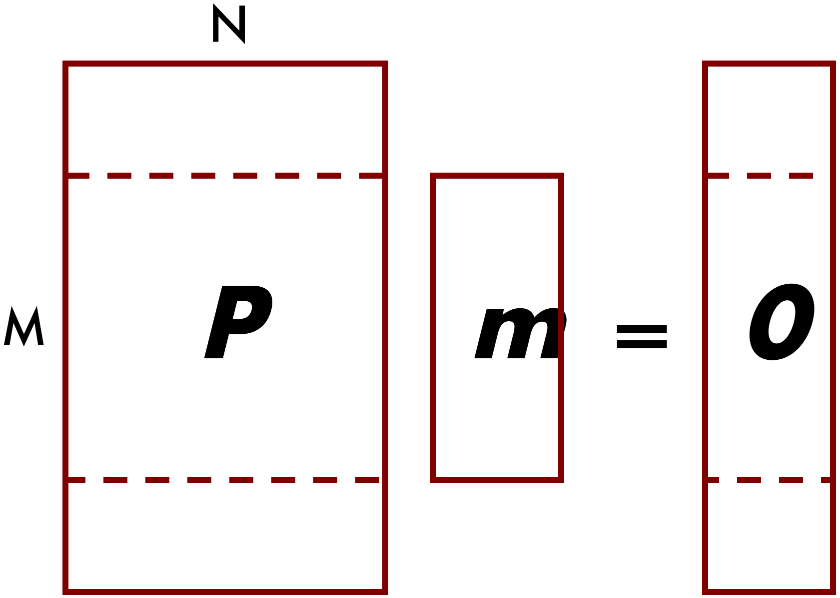
Homogenous linear system

$$\mathbf{P} \overset{\text{def}}{=} \begin{pmatrix} \mathrm{P}_1^T & \mathbf{0}^T & -u_1\overset{\text{1x4}}{\boxed{\mathrm{P}_1^T}} \\ \mathbf{0}^T & \mathrm{P}_1^T & -v_1\mathrm{P}_1^T \\ & \vdots & \\ \mathrm{P}_\mathrm{n}^T & \mathbf{0}^T & -u_n\mathrm{P}_\mathrm{n}^T \\ \mathbf{0}^T & \mathrm{P}_\mathrm{n}^T & -v_n\mathrm{P}_\mathrm{n}^T \end{pmatrix}_{\text{2n x 12}}$$

$$\boldsymbol{m} \overset{\text{def}}{=} \begin{pmatrix} \overset{\text{4x1}}{\boxed{\mathbf{m}_1^\mathrm{T}}} \\ \mathbf{m}_2^\mathrm{T} \\ \mathbf{m}_3^\mathrm{T} \end{pmatrix}_{\text{12x1}}$$

Homogeneous M x N Linear Systems

M=number of equations = 2n
N=number of unknown = 11



Rectangular system (M>N)

- 0 is always a solution
- To find non-zero solution

Minimize $|\mathbf{P}\ \mathbf{m}|^2$

under the constraint $|\mathbf{m}|^2 = 1$

Calibration Problem

$$\mathbf{P}\ \mathbf{m} = 0$$

- How do we solve this homogenous linear system?
- Via SVD decomposition!

Calibration Problem

$$\boxed{\mathbf{P}}\,\mathbf{m} = 0$$

$$\big\downarrow$$

SVD decomposition of P

$$\boxed{\mathbf{U}_{2n\times 12}\;\mathbf{D}_{12\times 12}\;\mathbf{V}^{T}{}_{12\times 12}}$$

Last column of V gives $\boldsymbol{m}$

Why? See pag 592 of HZ

$$\begin{array}{c}\boldsymbol{m}\\ \big\downarrow\\ M\end{array}$$

$$\boldsymbol{m} \overset{\text{def}}{=} \begin{pmatrix}\mathbf{m}_1^{T}\\ \mathbf{m}_2^{T}\\ \mathbf{m}_3^{T}\end{pmatrix}$$

Extracting camera parameters

$$M = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix} \rho$$

Extracting camera parameters

See [FP], Sec. 1.3.1

$$\frac{M}{\rho} = \left(\underbrace{\begin{matrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{matrix}}_{A} \quad \underbrace{\begin{matrix} \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ t_z \end{matrix}}_{\mathbf{b}} \right) = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} \alpha & -\alpha\cot\theta & \mathrm{u}_o \\ 0 & \dfrac{\beta}{\sin\theta} & \mathrm{v}_o \\ 0 & 0 & 1 \end{bmatrix}$$

Box 1

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

Estimated values

Intrinsic

$$\rho = \frac{\pm 1}{|\mathbf{a}_3|} \qquad u_o = \rho^2(\mathbf{a}_1 \cdot \mathbf{a}_3)$$

$$\mathrm{v}_o = \rho^2(\mathbf{a}_2 \cdot \mathbf{a}_3)$$

$$\cos\theta = \frac{(\mathbf{a}_1 \times \mathbf{a}_3)\cdot(\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_1 \times \mathbf{a}_3| \cdot |\mathbf{a}_2 \times \mathbf{a}_3|}$$

Extracting camera parameters

See [FP], Sec. 1.3.1

$$\frac{M}{\rho} = \begin{pmatrix} \underbrace{\begin{matrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \frac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{matrix}}_{A} & \underbrace{\begin{matrix} \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \frac{\beta}{\sin\theta} t_y + v_0 t_z \\ t_z \end{matrix}}_{\mathbf{b}} \end{pmatrix} = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} \alpha & -\alpha \cot\theta & u_o \\ 0 & \frac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{bmatrix}$$

Box 1

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

Estimated values

Intrinsic

$$\alpha = \rho^2 \left| \mathbf{a}_1 \times \mathbf{a}_3 \right| \sin\theta$$

$$\beta = \rho^2 \left| \mathbf{a}_2 \times \mathbf{a}_3 \right| \sin\theta$$

Theorem (Faugeras, 1993)

Let $\mathcal{M} = (\,\mathcal{A} \quad \boldsymbol{b}\,)$ be a 3×4 matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

Extracting camera parameters

See [FP], Sec. 1.3.1

$$\frac{M}{\rho} = \left(\underbrace{\begin{matrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \frac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{matrix}}_{A} \quad \underbrace{\begin{matrix} \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \frac{\beta}{\sin\theta} t_y + v_0 t_z \\ t_z \end{matrix}}_{\mathbf{b}} \right) = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} \alpha & -\alpha \cot\theta & u_o \\ 0 & \frac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{bmatrix}$$

Box 1

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

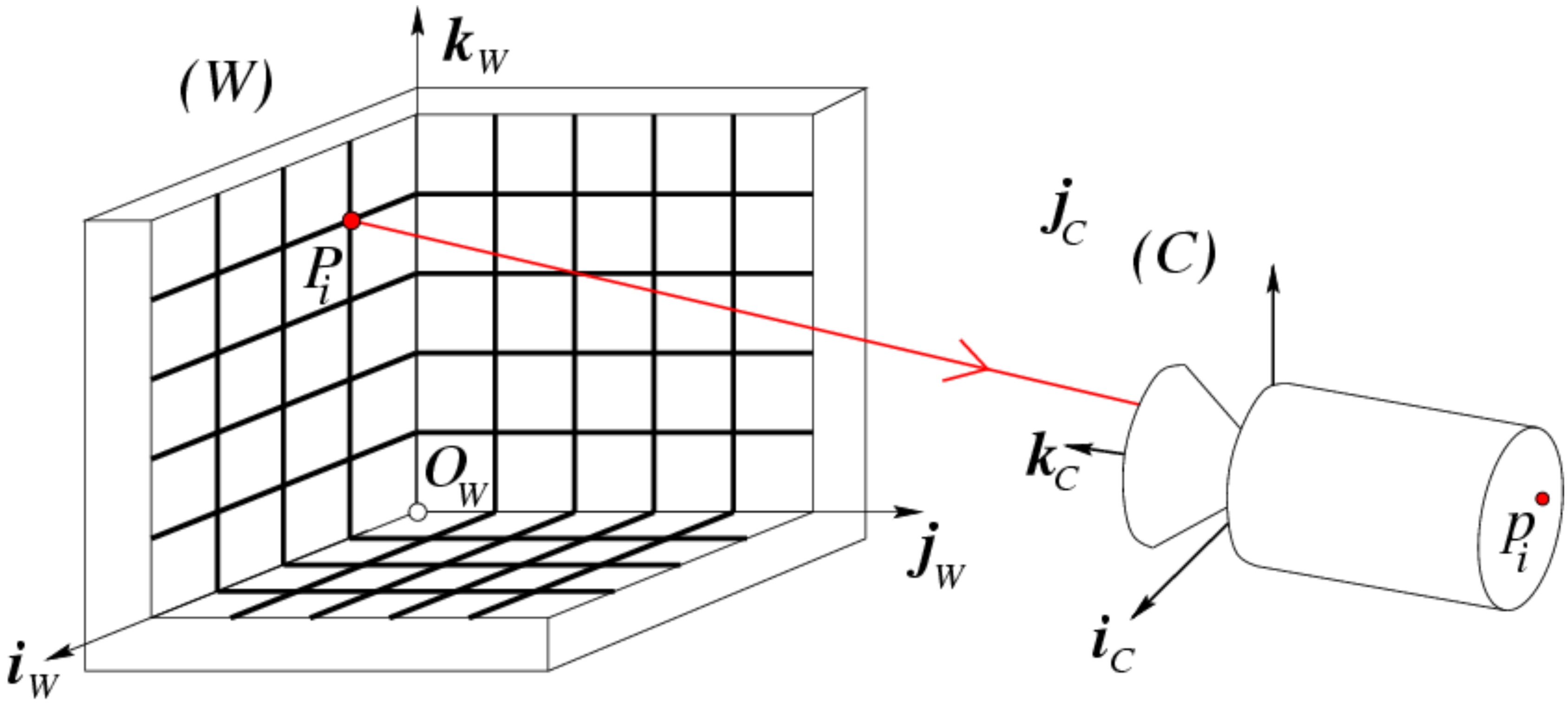
Estimated values

Extrinsic

$$\mathbf{r}_1 = \frac{(\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_2 \times \mathbf{a}_3|} \qquad \mathbf{r}_3 = \frac{\pm \mathbf{a}_3}{|\mathbf{a}_3|}$$

$$\mathbf{r}_2 = \mathbf{r}_3 \times \mathbf{r}_1 \qquad \mathrm{T} = \rho\, \mathrm{K}^{-1}\mathbf{b}$$

Degenerate cases



- P_i's cannot lie on the same plane!
- Points cannot lie on the intersection curve of two quadric surfaces [FP] section 1.3

Lecture 3

Camera Calibration

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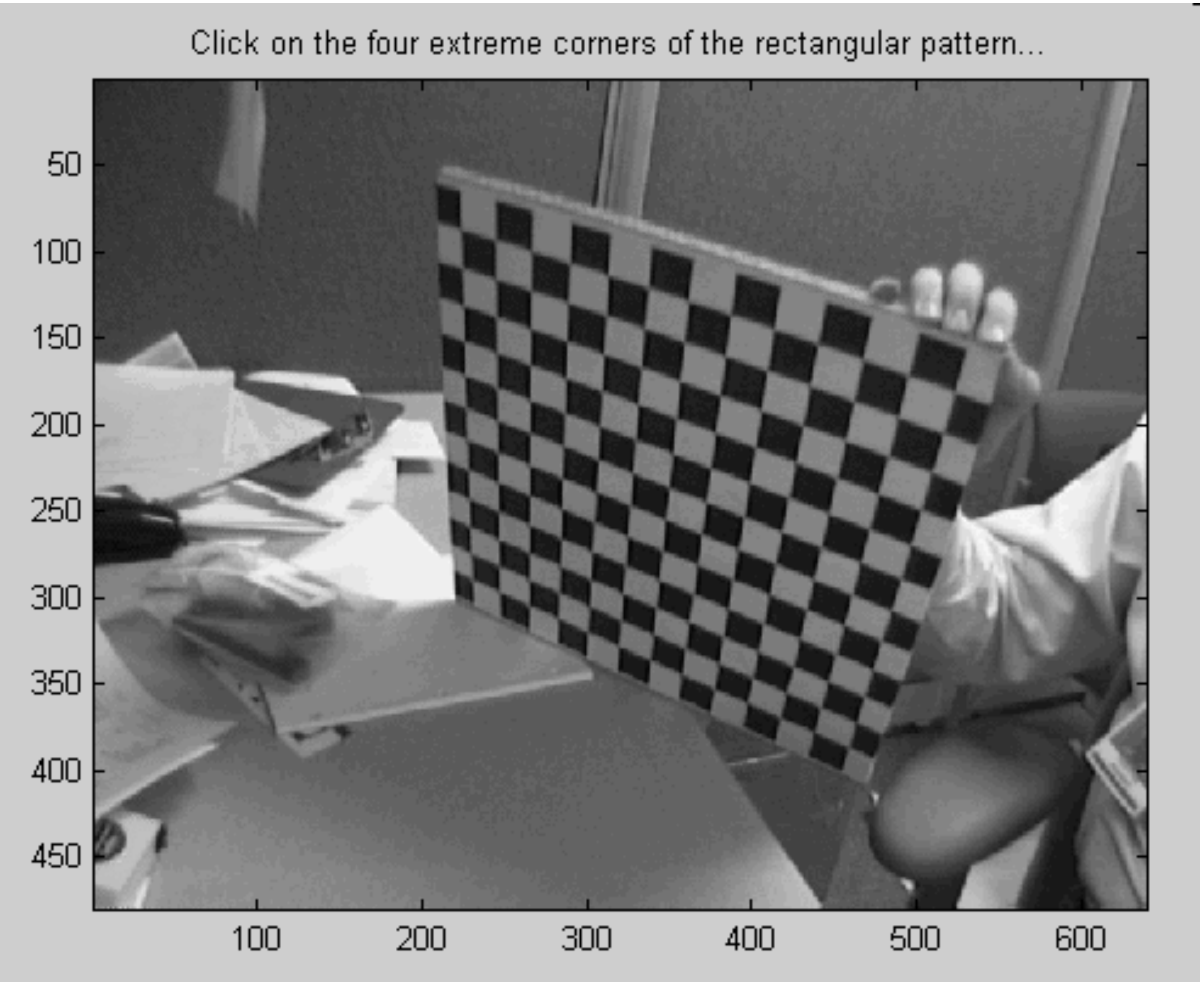
Camera Calibration video from Matlab

https://www.youtube.com/watch?v=g8SyzR0jZOA

Calibration Procedure

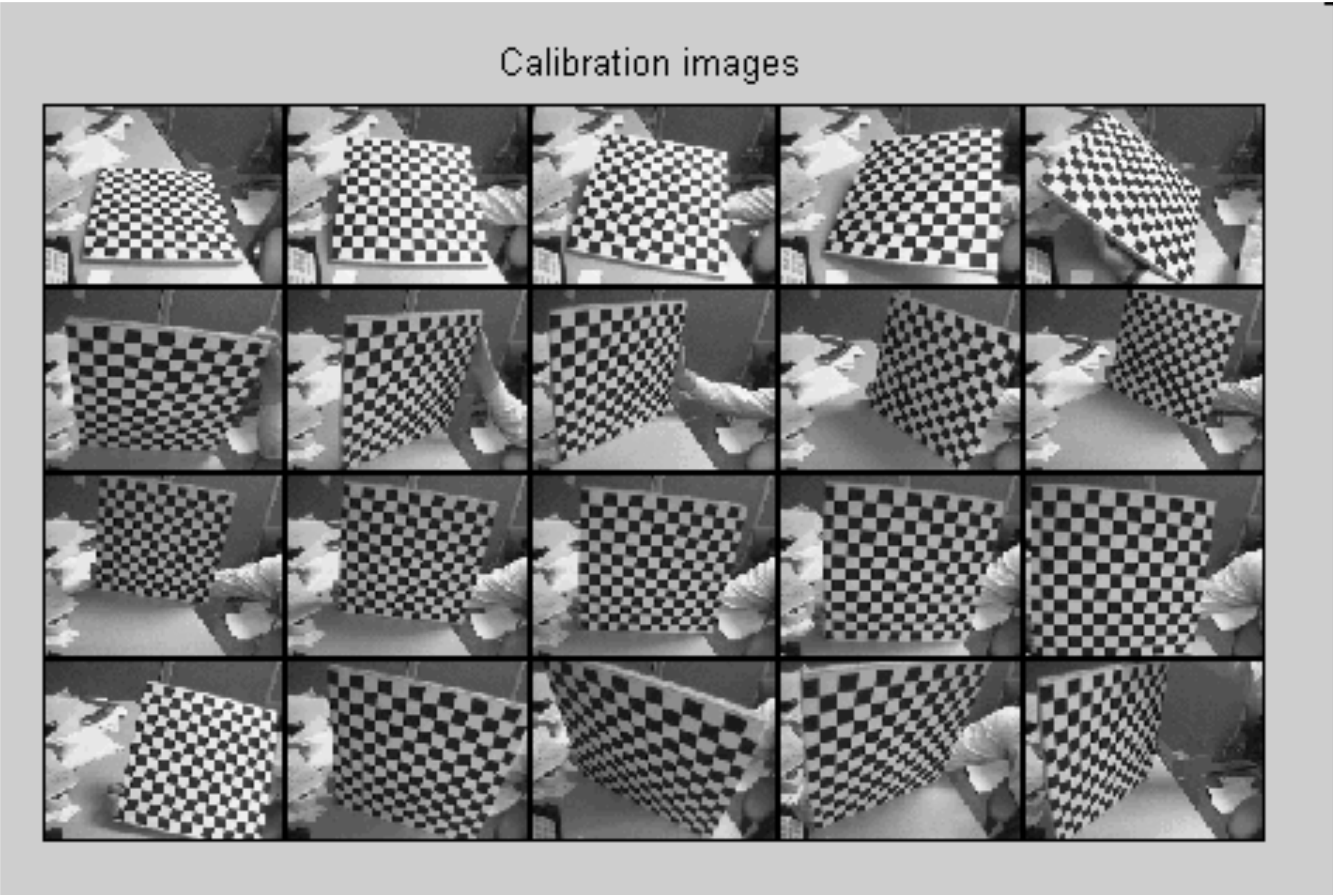
Camera Calibration Toolbox for Matlab
J. Bouguet – [1998-2000]

http://www.vision.caltech.edu/bouguetj/calib_doc/index.html#examples

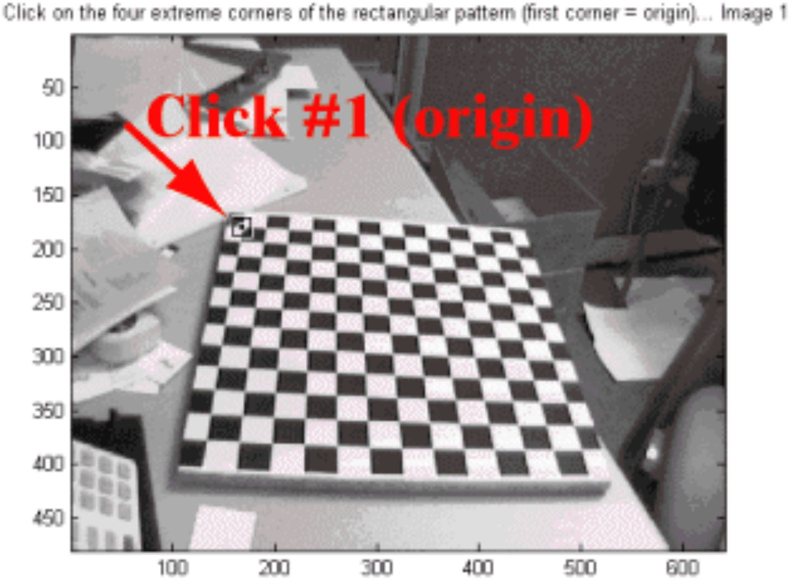


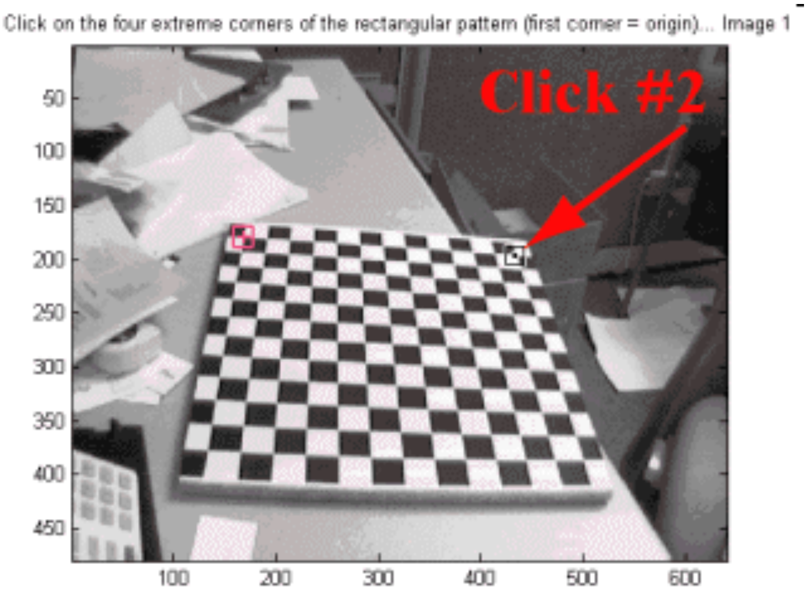
Click on the four extreme corners of the rectangular pattern...

Calibration Procedure



Calibration Procedure


Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1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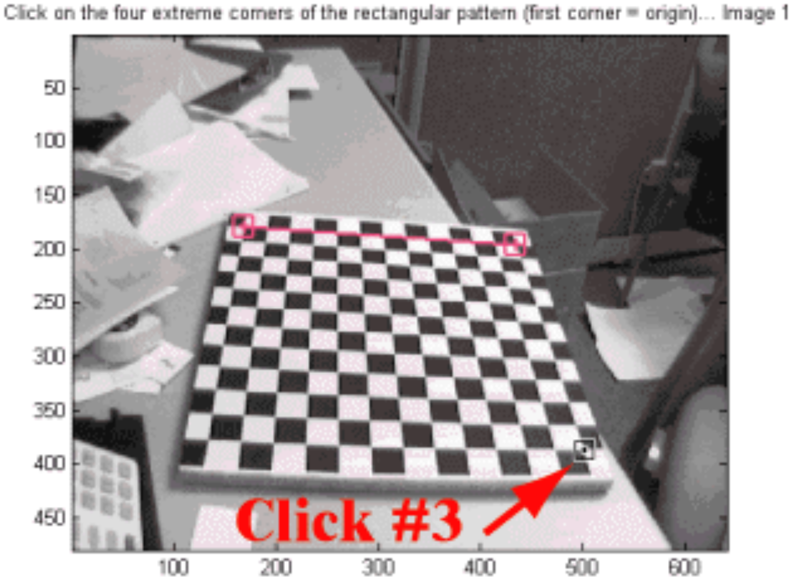
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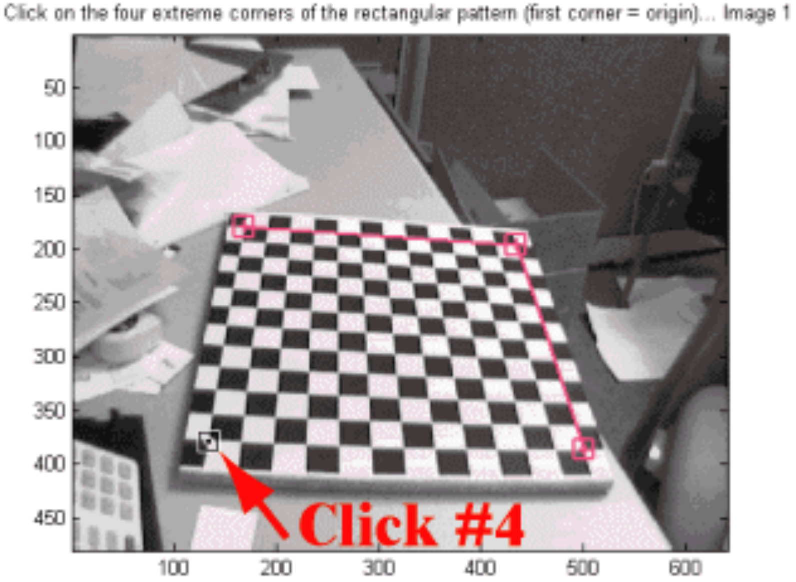
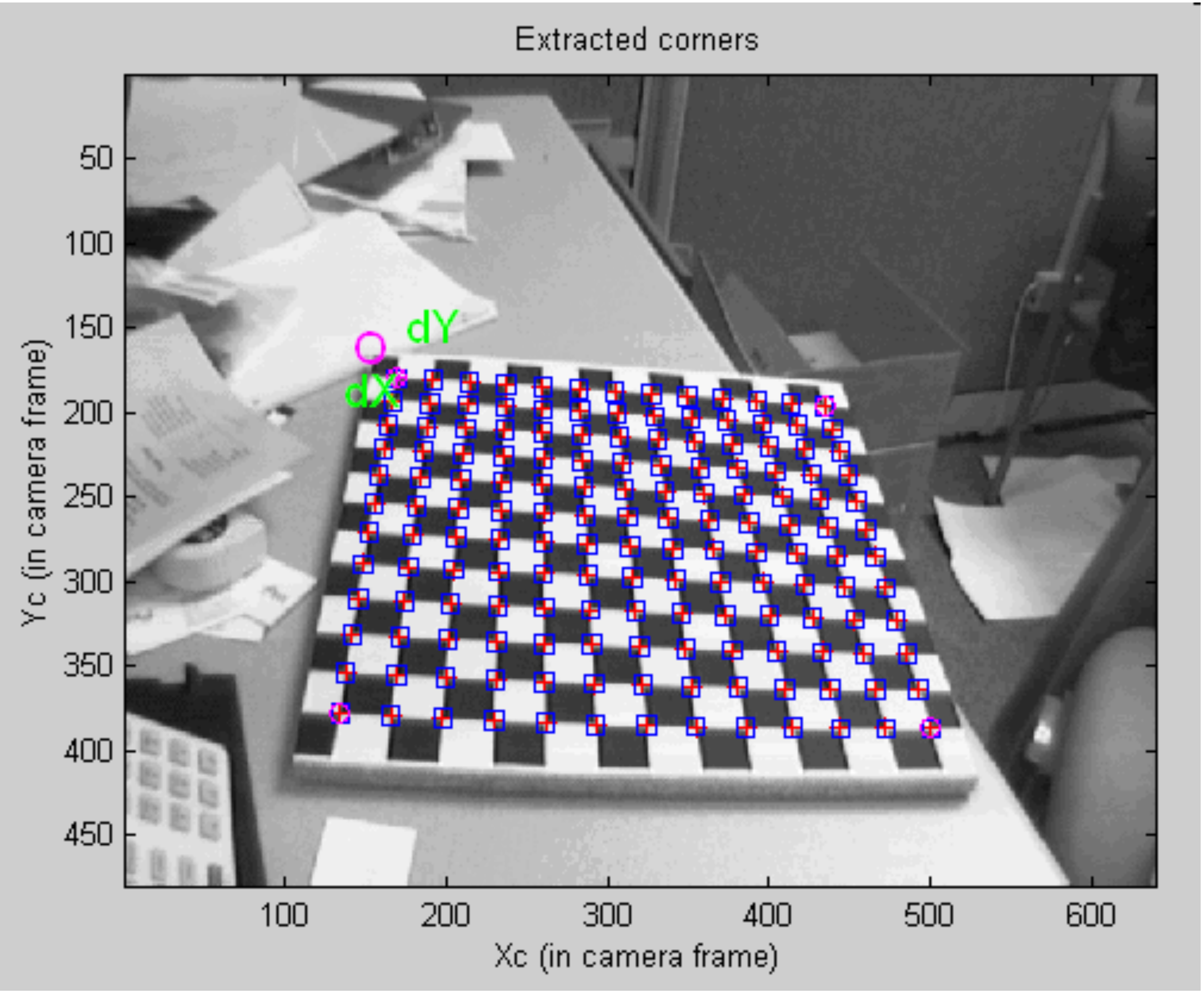
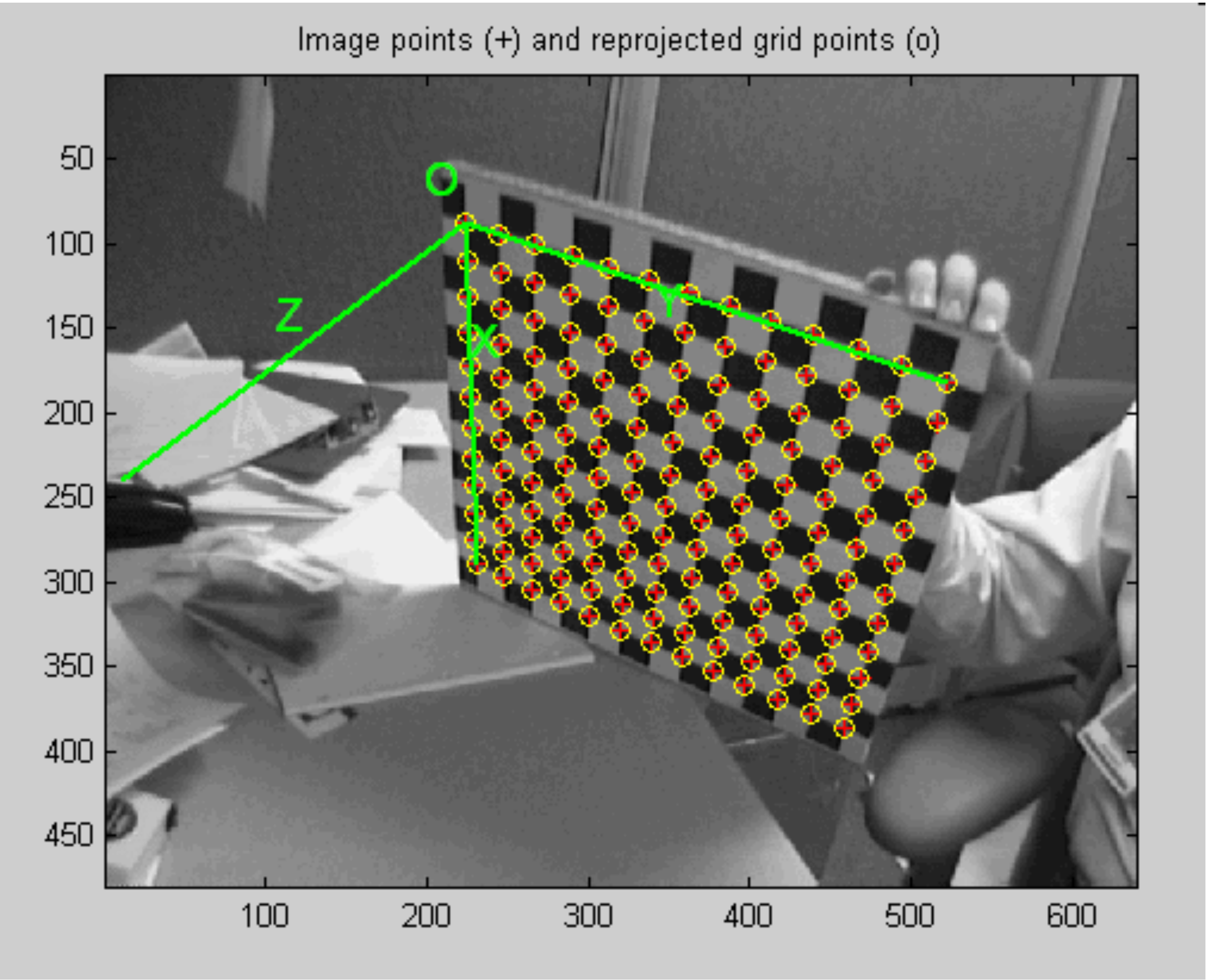

Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1

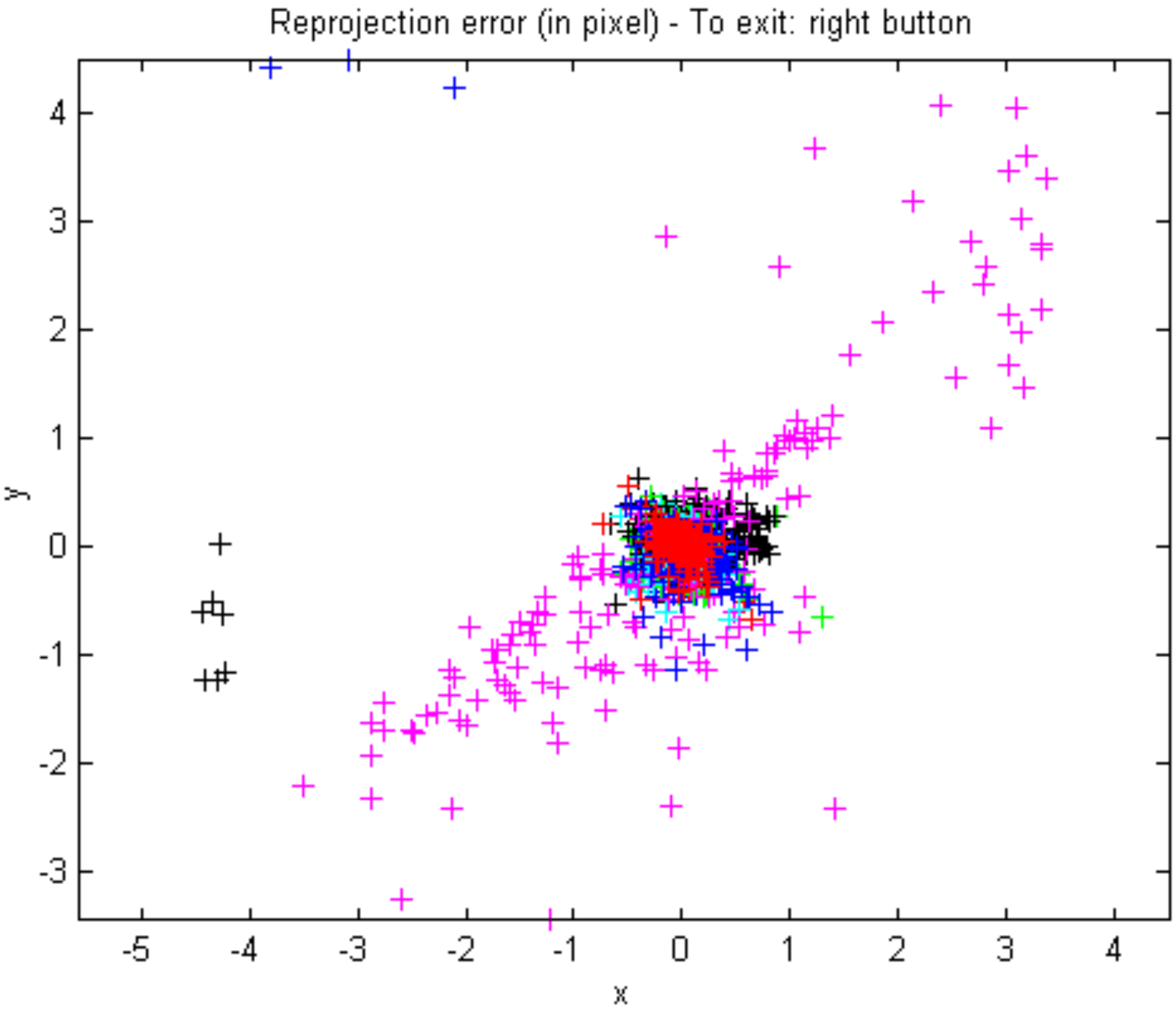
Calibration Procedure



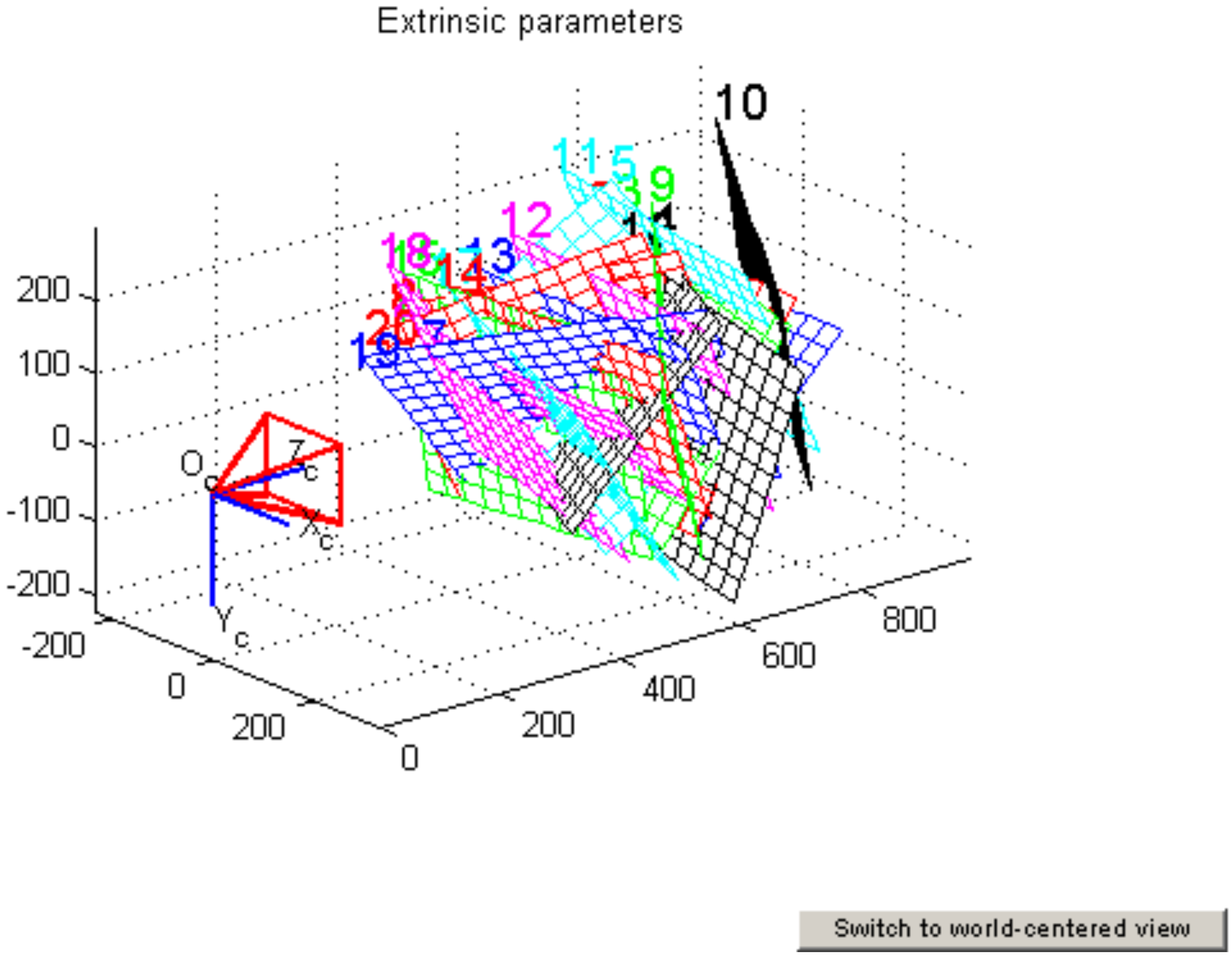
Calibration Procedure



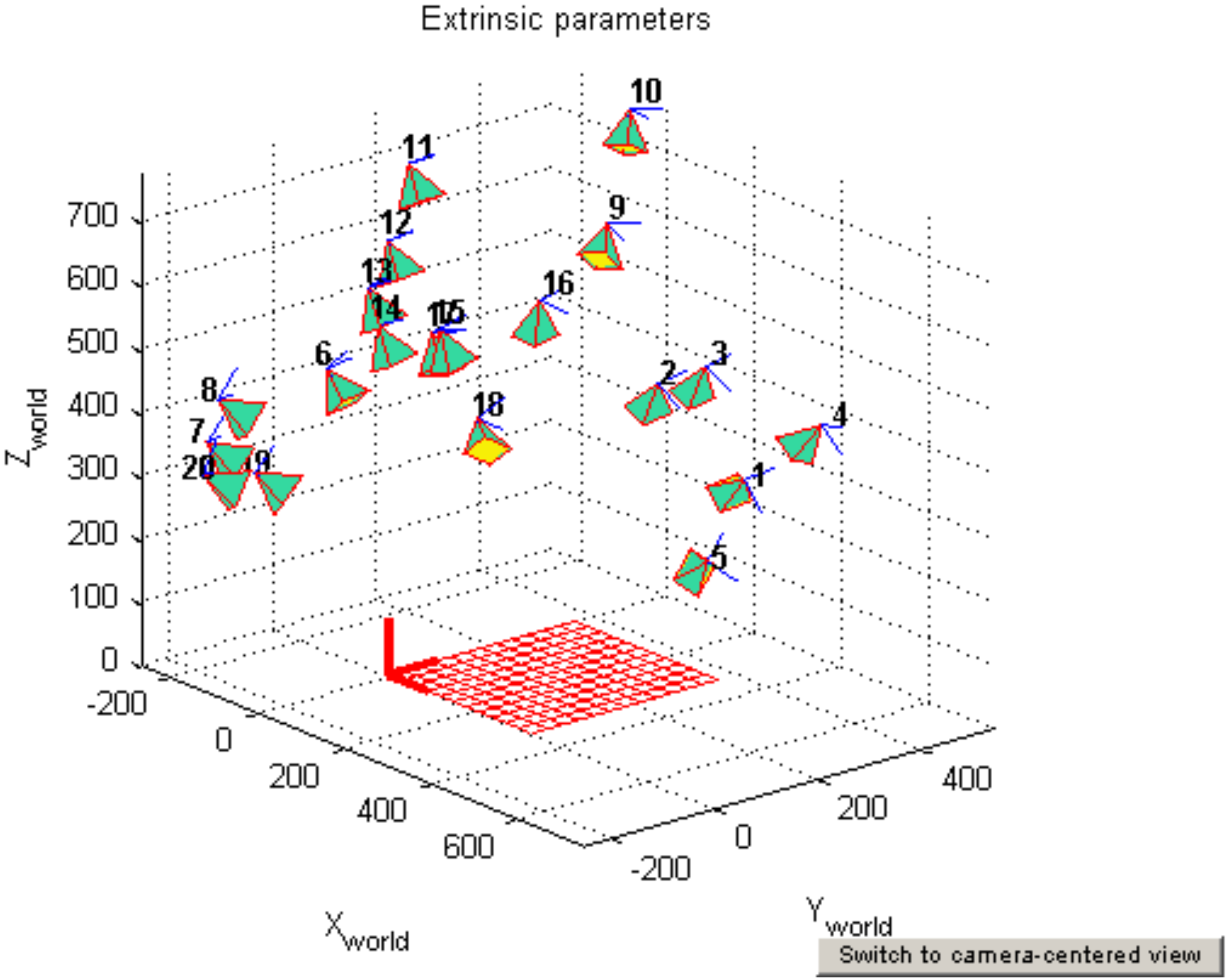
Calibration Procedure



Calibration Procedure



Calibration Procedure



Next lecture

- Single view reconstruction

Eigenvalues and Eigenvectors

Eigendecomposition

$$A = S\Lambda\, S^{-1} = S \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \lambda_N \end{bmatrix} S^{-1}$$

Eigenvectors of A are columns of S

$$S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_N \end{bmatrix}$$

Singular Value decomposition

$$A = U\,\Sigma\,V^{-1} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

U, V = orthogonal matrix

$$\sigma_i = \sqrt{\lambda_i}$$

σ = singular value

λ = eigenvalue of $A^t A$