Lecture 2

Camera Models

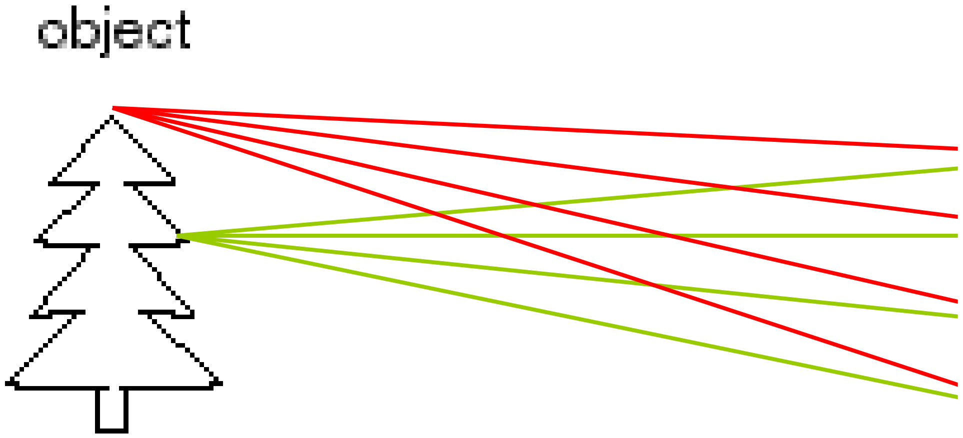
- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras

Reading:	[FP] Chapter 1, "Geometric Camera Models"
	[HZ] Chapter 6 "Camera Models"

Some slides in this lecture are courtesy to Profs. J. Ponce, S. Seitz, F-F Li

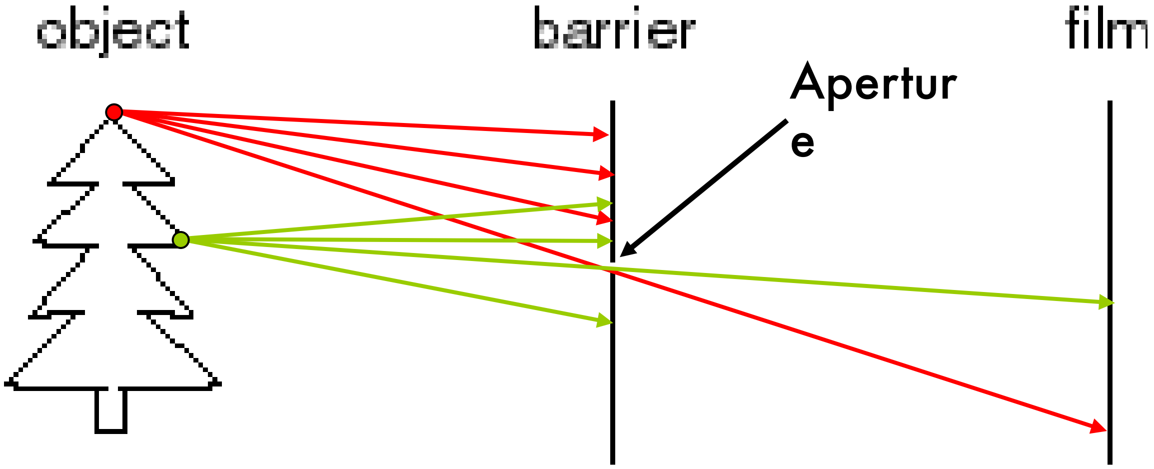
Silvio Savarese

How do we see the world?



- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?

Pinhole camera



- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture**

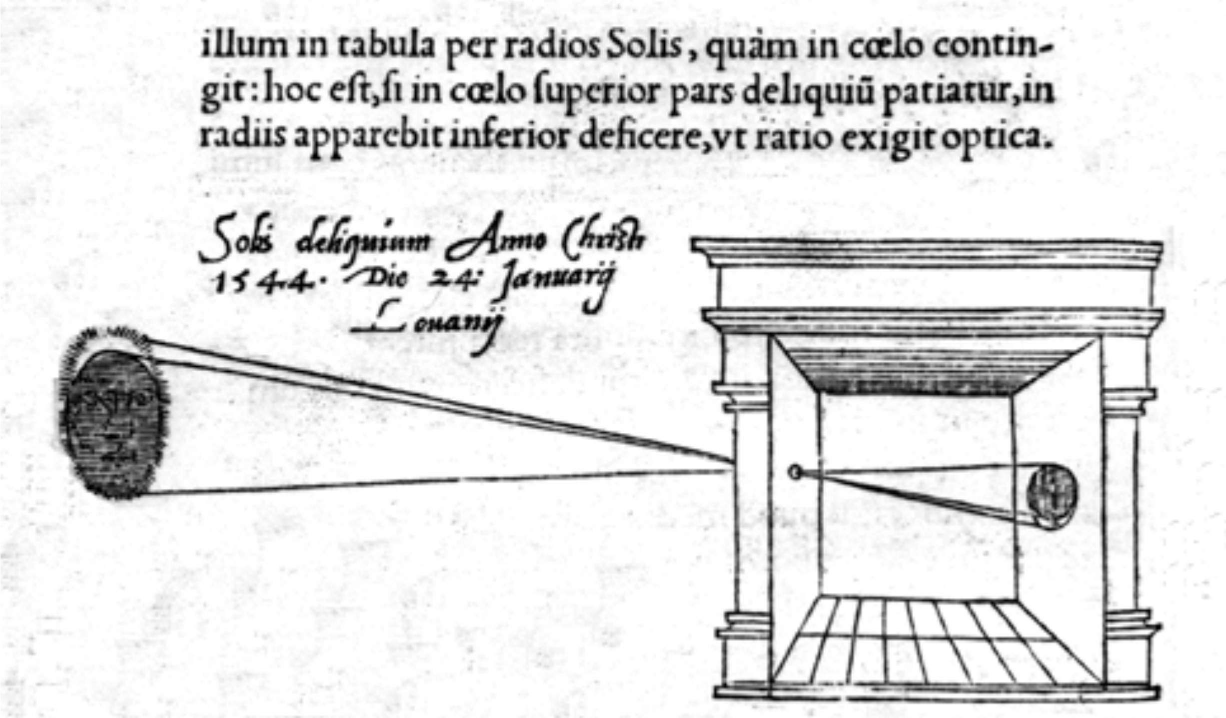
Some history...

Milestones:

- Leonardo da Vinci (1452-1519):

first record of camera *obscura (1502)*

https://www.youtube.com/watch?v=LutludRhm10



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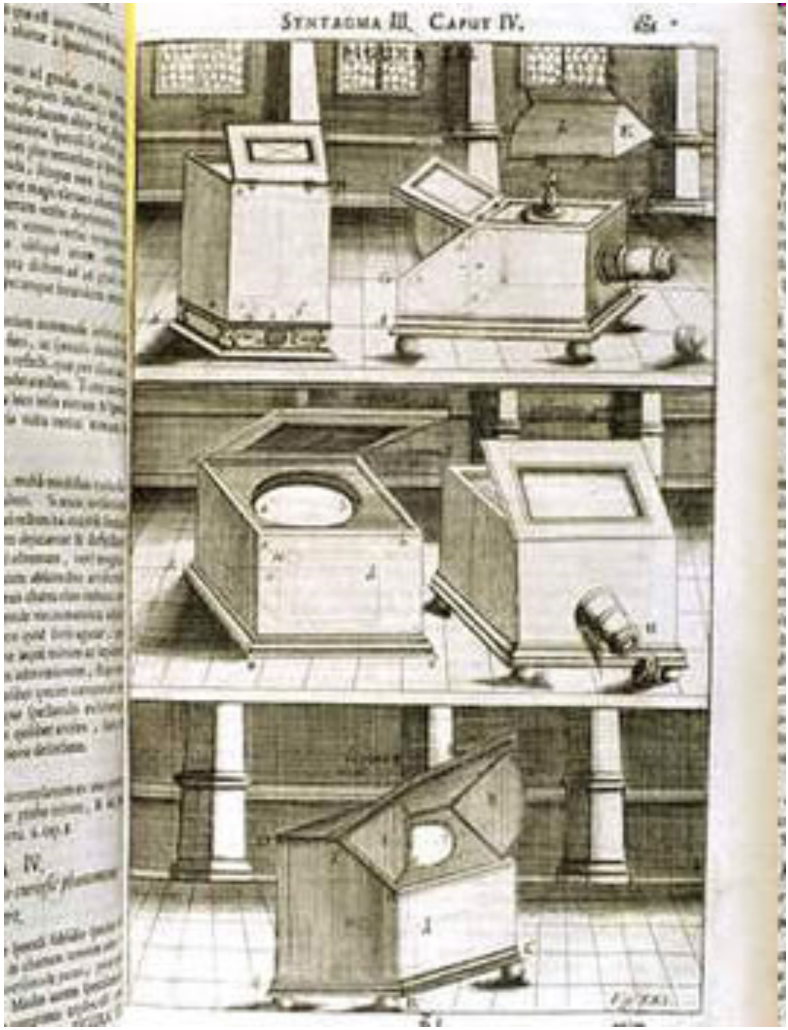
Some history...

Milestones:

- Leonardo da Vinci (1452-1519): first record of camera *obscura*

- Johann Zahn (1685): first portable camera



Some history...

Milestones:

- Leonardo da Vinci (1452-1519): first record of camera *obscura*

- Johann Zahn (1685): first portable camera

- Joseph Nicéphore Niépce (1822): first photo - birth of photography

Photography (Niépce, "La Table Servie," 1822)

Some history...

Milestones:

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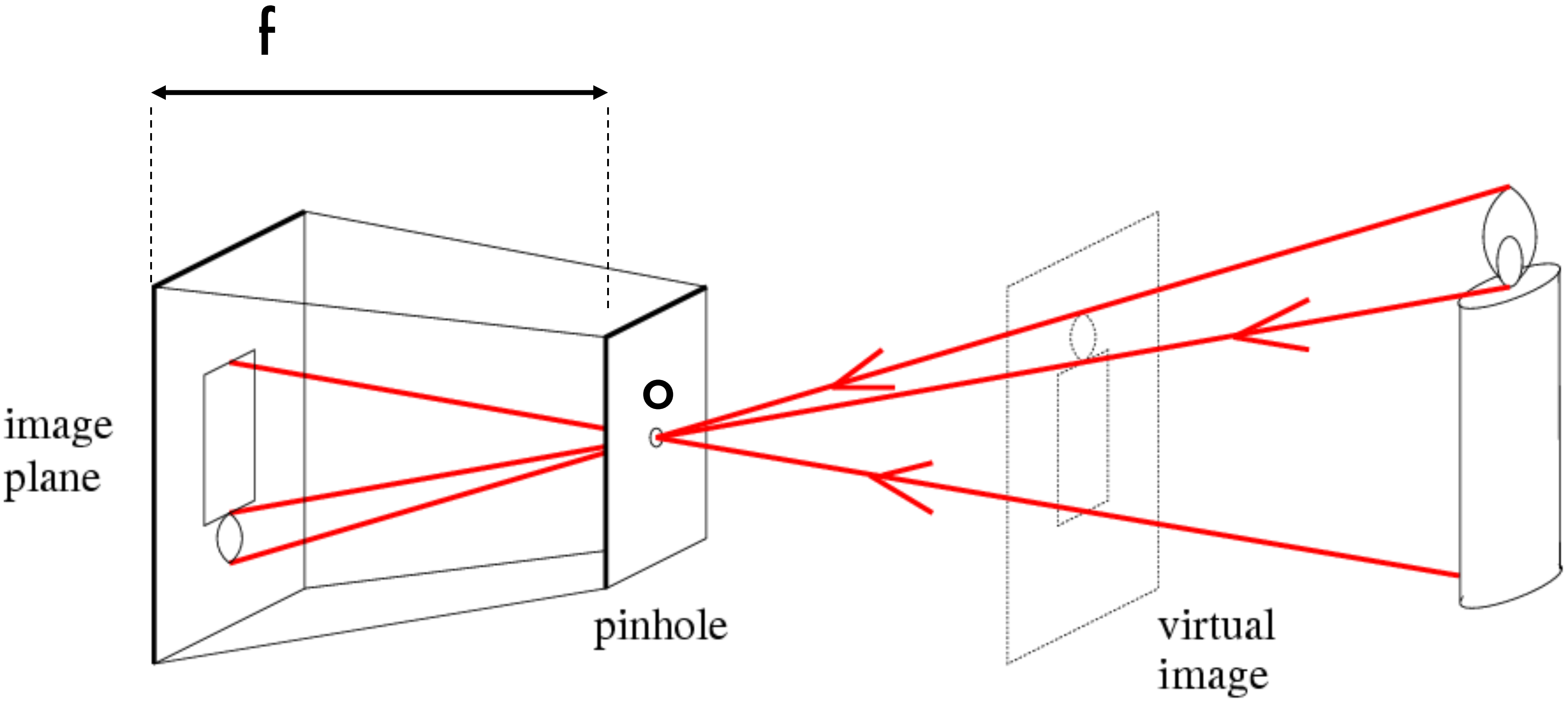
- Daguerréotypes (1839)
- Photographic Film (Eastman, 1889)
- Cinema (Lumière Brothers, 1895)
- Color Photography (Lumière Brothers, 1908)

Photography (Niépce, "La Table Servie," 1822)

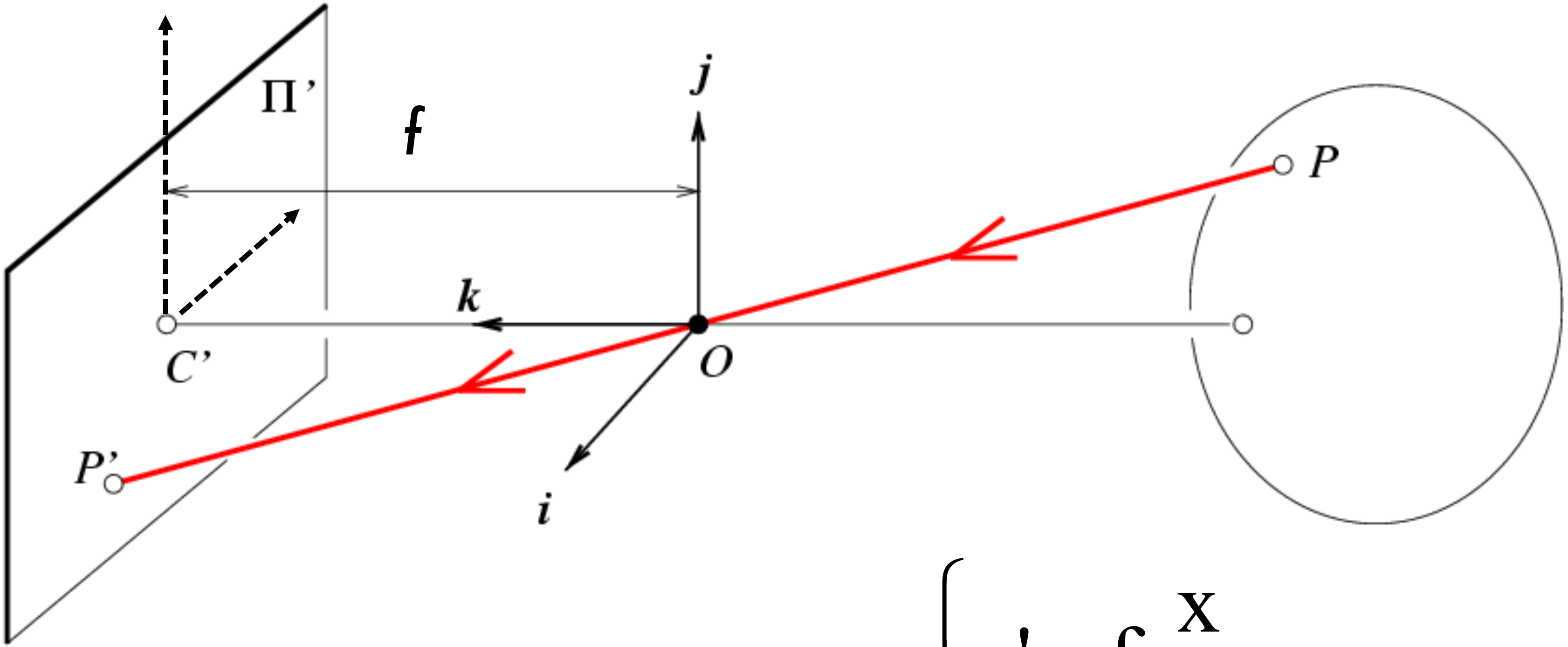
Pinhole camera



f = focal length

o = aperture = pinhole = center of the camera

Pinhole camera



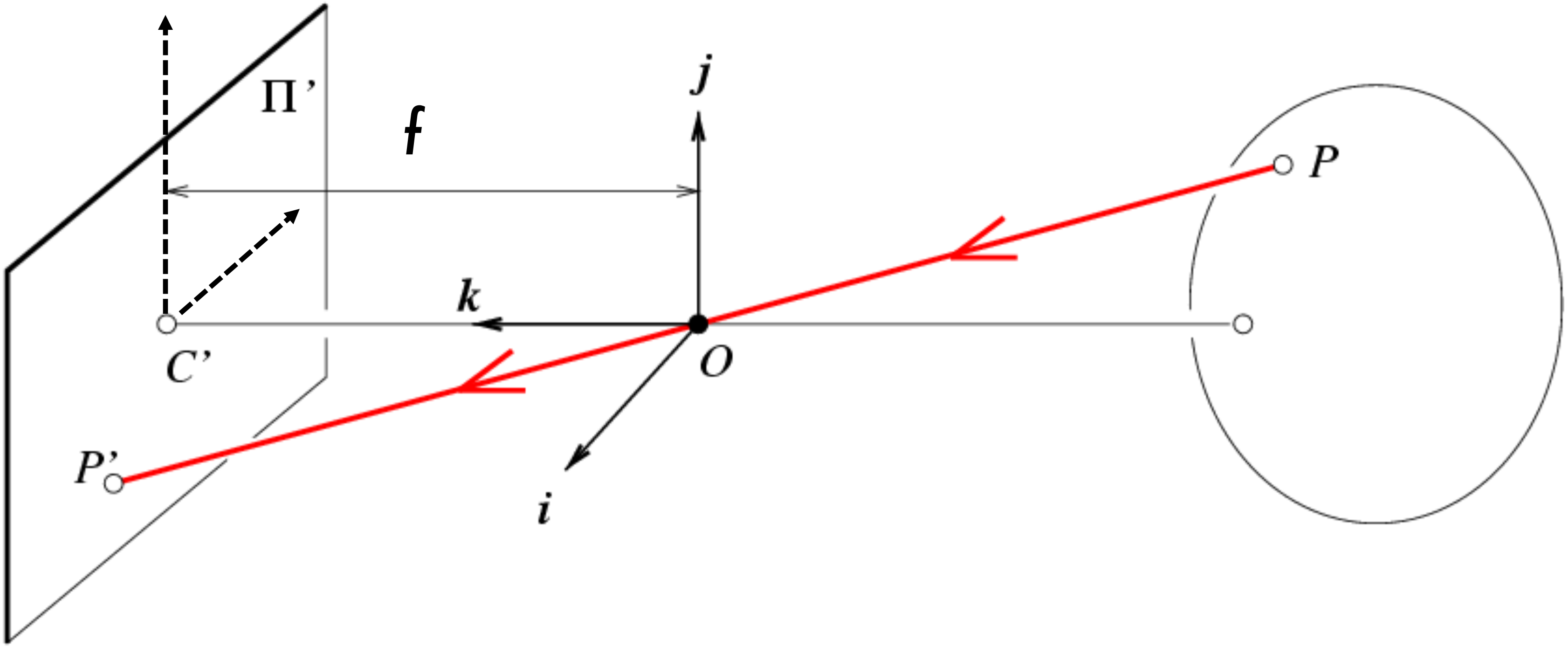
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

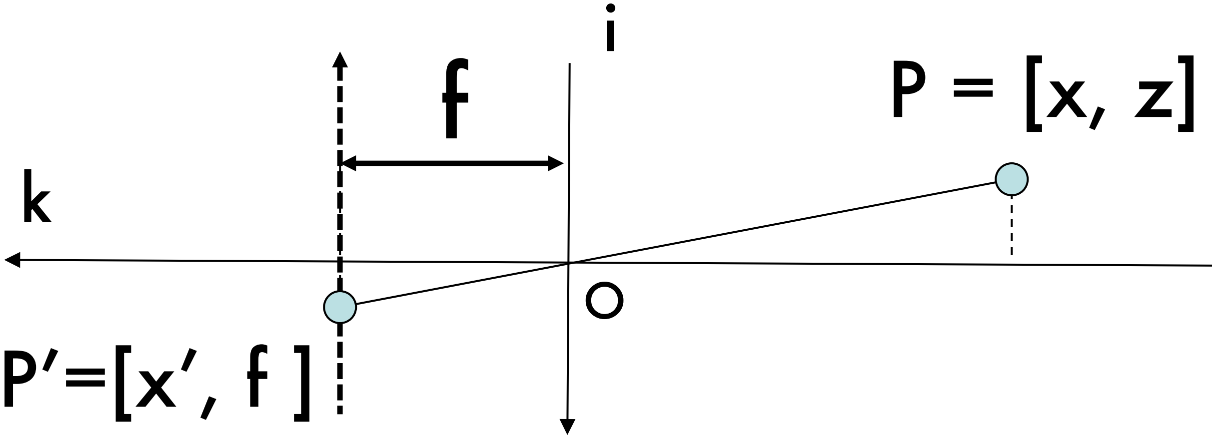
$$\begin{cases} x' = f \dfrac{x}{z} \\ y' = f \dfrac{y}{z} \end{cases}$$

[Eq. 1]

Derived using similar triangles

Pinhole camera



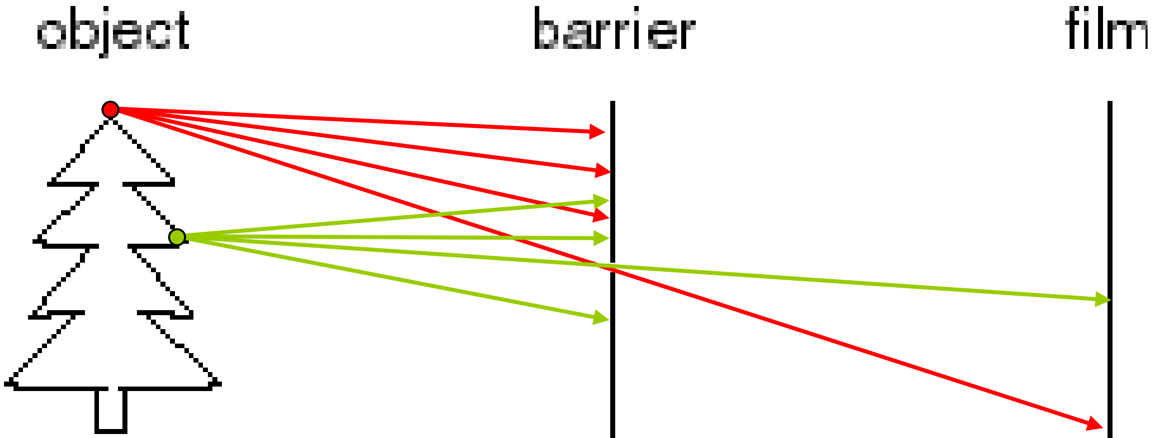


[Eq. 2]

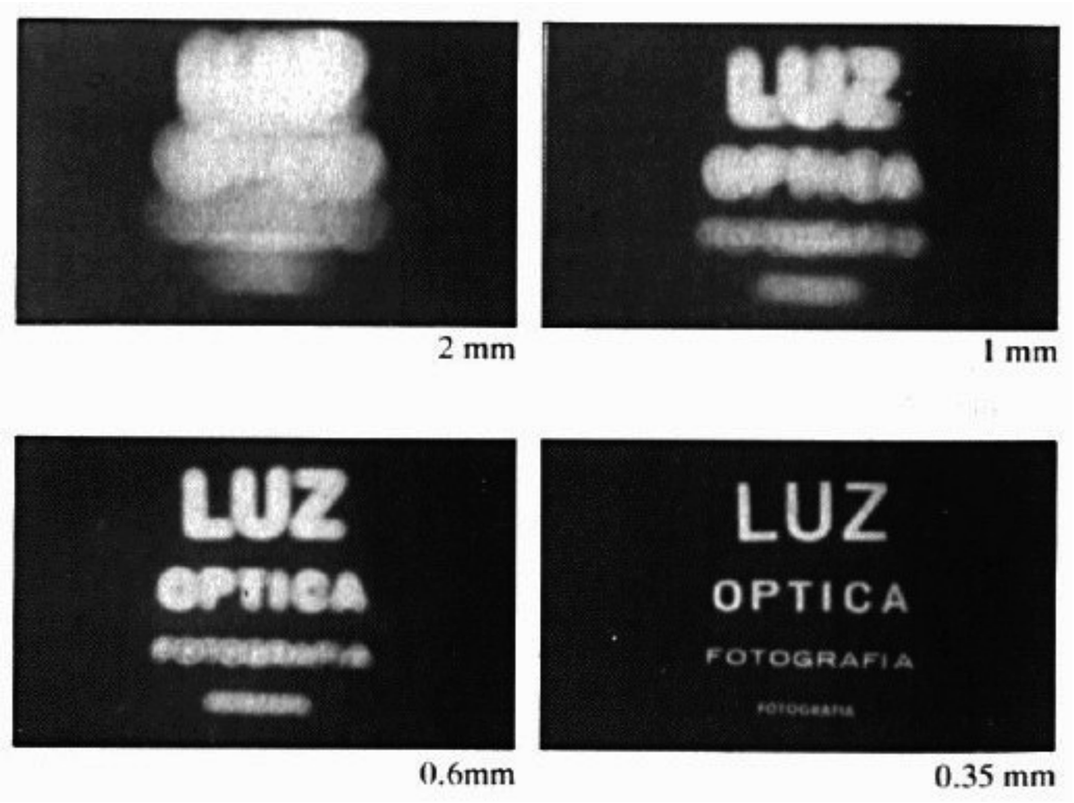
$$\frac{x'}{f} = \frac{x}{z}$$

Pinhole camera

Is the size of the aperture important?



Shrinking aperture size

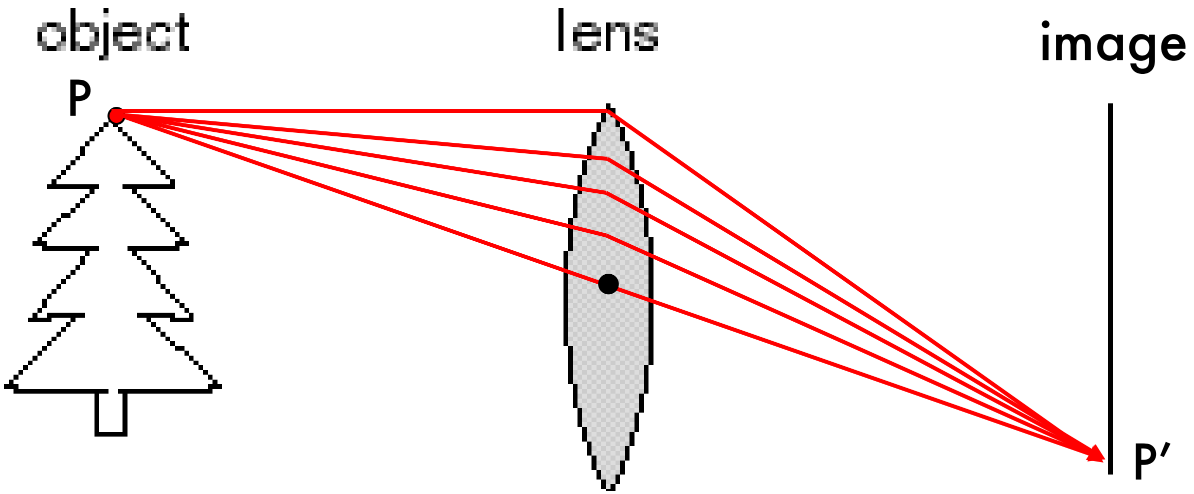


-What happens if the aperture is too small?

-Less light passes through

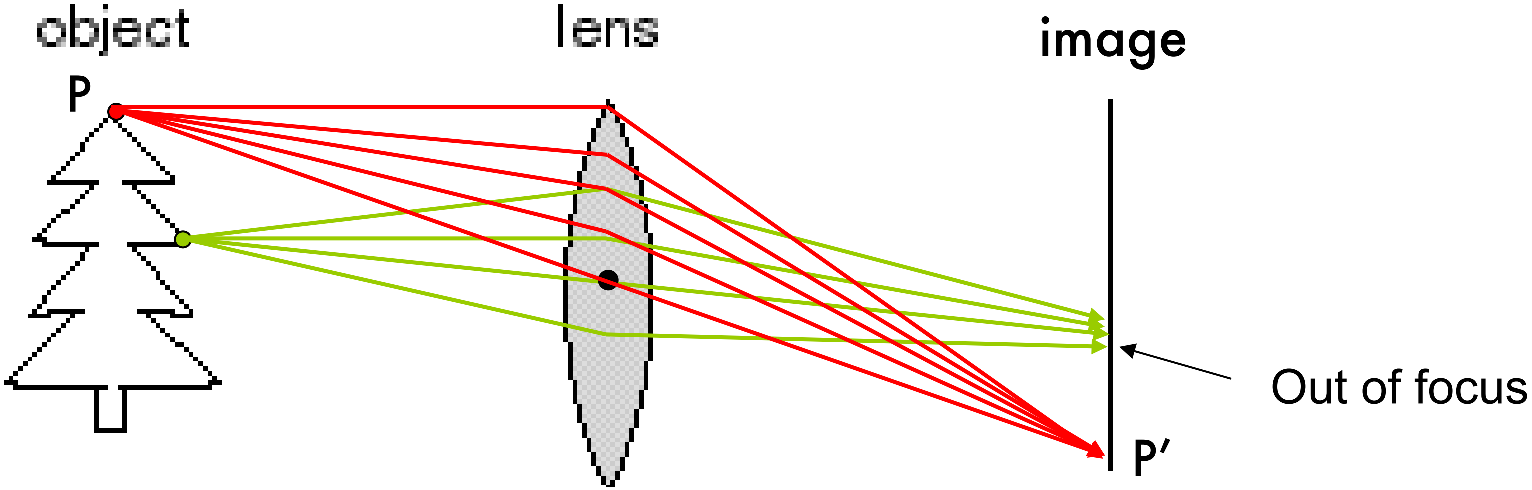
Adding lenses!

Cameras & Lenses



- A lens focuses light onto the film

Cameras & Lenses

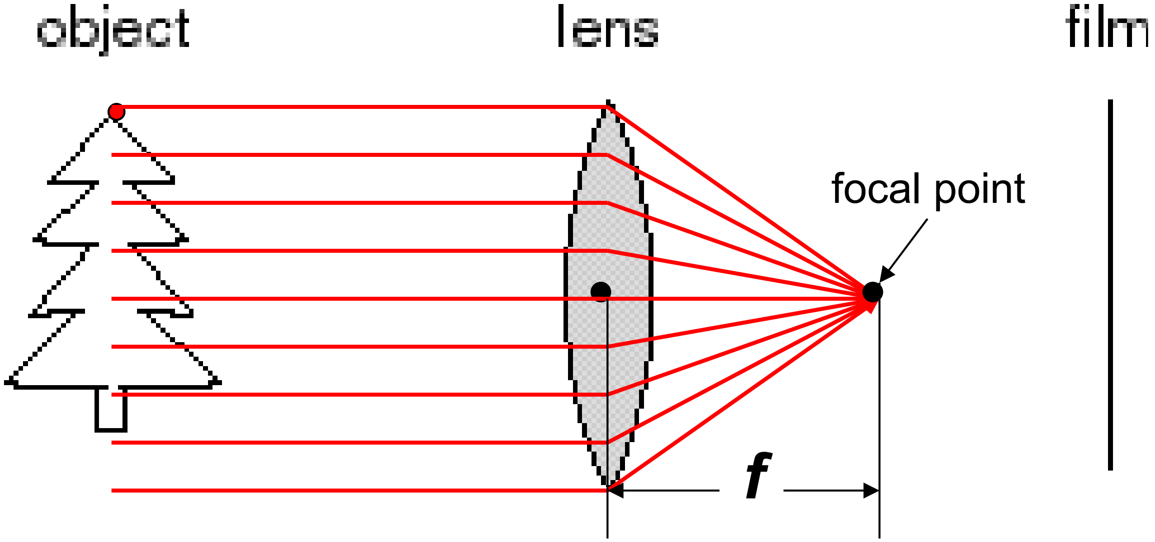


- A lens focuses light onto the film
  - There is a specific distance at which objects are "in focus"
  - Related to the concept of depth of field

Cameras & Lenses

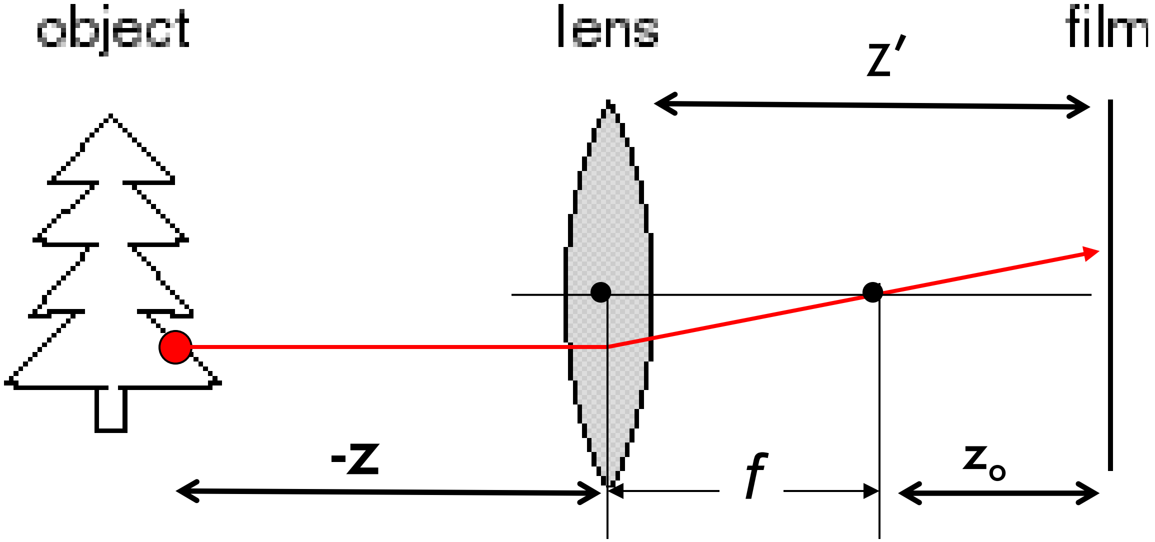
- A lens focuses light onto the film
  - There is a specific distance at which objects are "in focus"
  - Related to the concept of depth of field

Cameras & Lenses



- A lens focuses light onto the film
  - All rays parallel to the optical (or principal) axis converge to one point (the *focal point*) on a plane located at the *focal length f* from the center of the lens.
  - Rays passing through the center are not deviated

Paraxial refraction model



From Snell's law:

[Eq. 3]

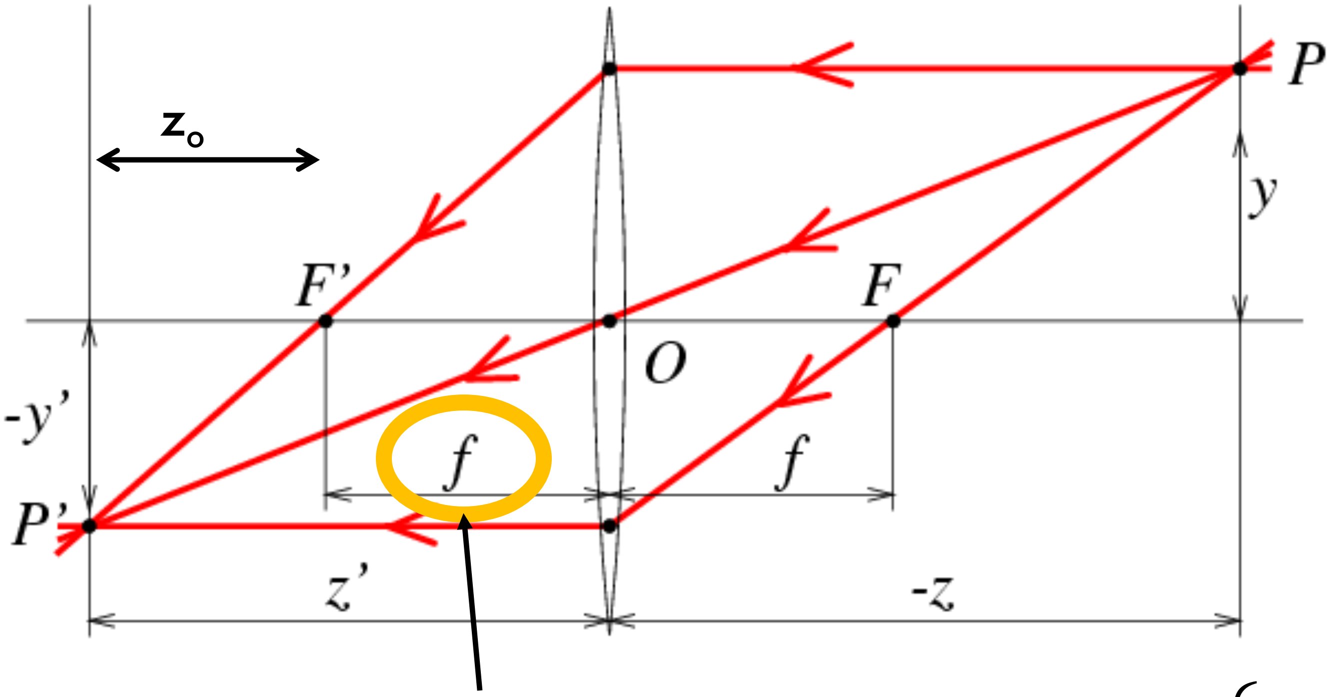
$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$

[Eq. 4]

$$\boxed{\begin{array}{l} z' = f + z_o \\ f = \dfrac{R}{2(n-1)} \end{array}}$$

Thin Lenses

[FP] sec 1.1, page 8.



Focal length

$$z' = f + z_o$$

$$f = \frac{R}{2(n-1)}$$

Snell's law:

$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$ ⇨

Small angles:

$n_1 \alpha_1 \approx n_2 \alpha_2$

$n_1 = \mathrm{n}$ (lens)

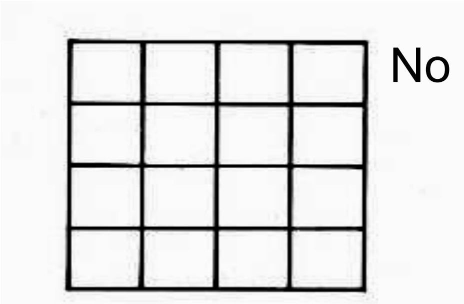
$n_1 = 1$ (air)

⇨

$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$

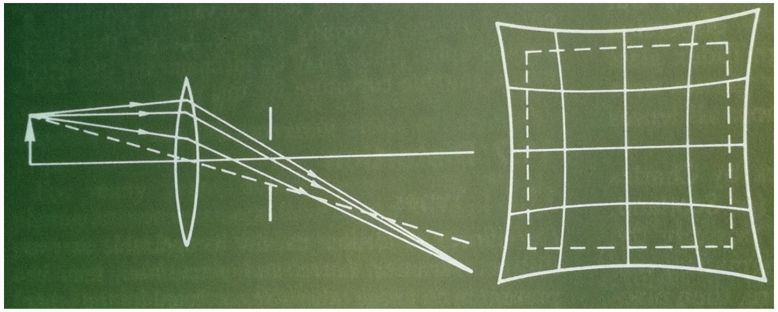
Issues with lenses: Radial Distortion

- Deviations are most noticeable for rays that pass through the edge of the lens



No distortion

Pin cushion



Barrel (fisheye lens)

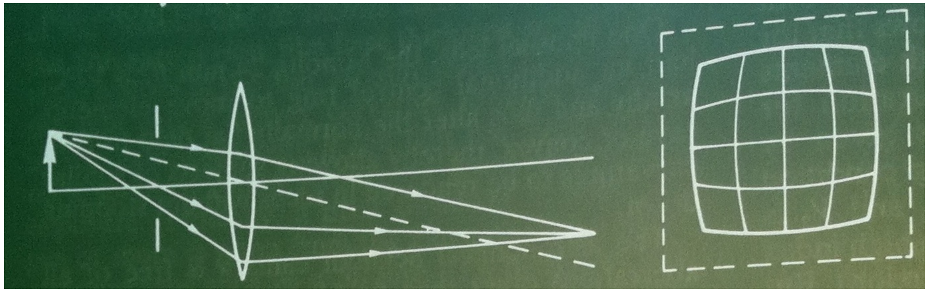


Image magnification decreases with distance from the optical axis

Lecture 2

Camera Models

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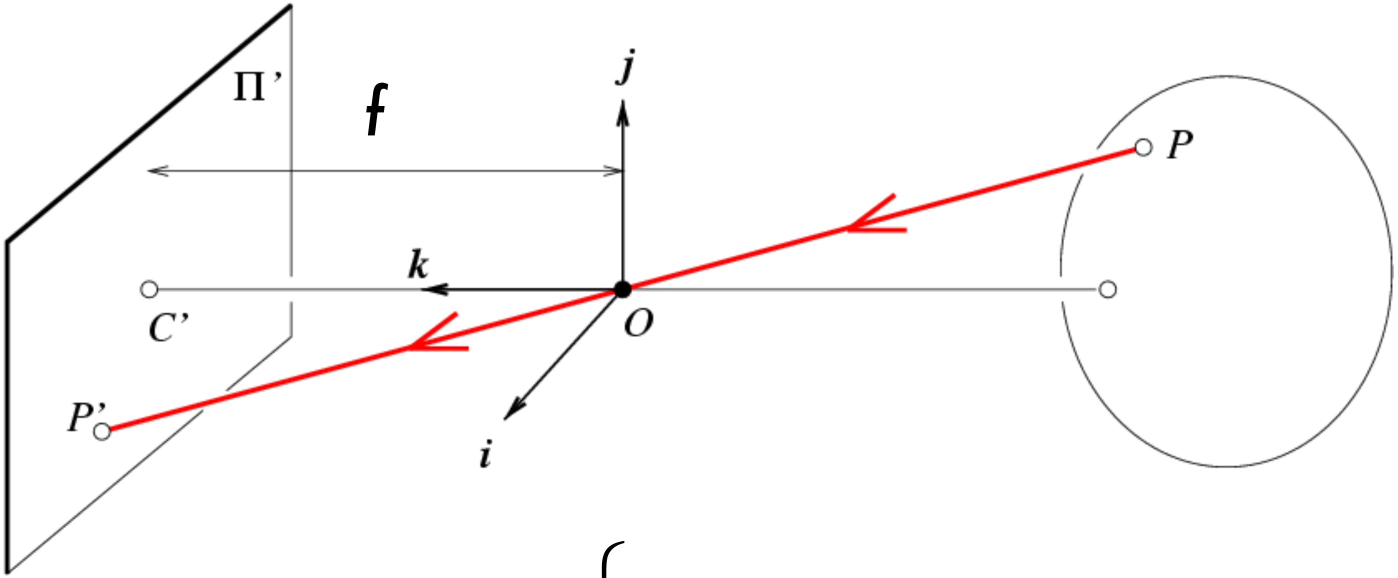
- Cameras & lenses
- The geometry of pinhole cameras
    - Intrinsic
    - Extrinsic

Pinhole camera



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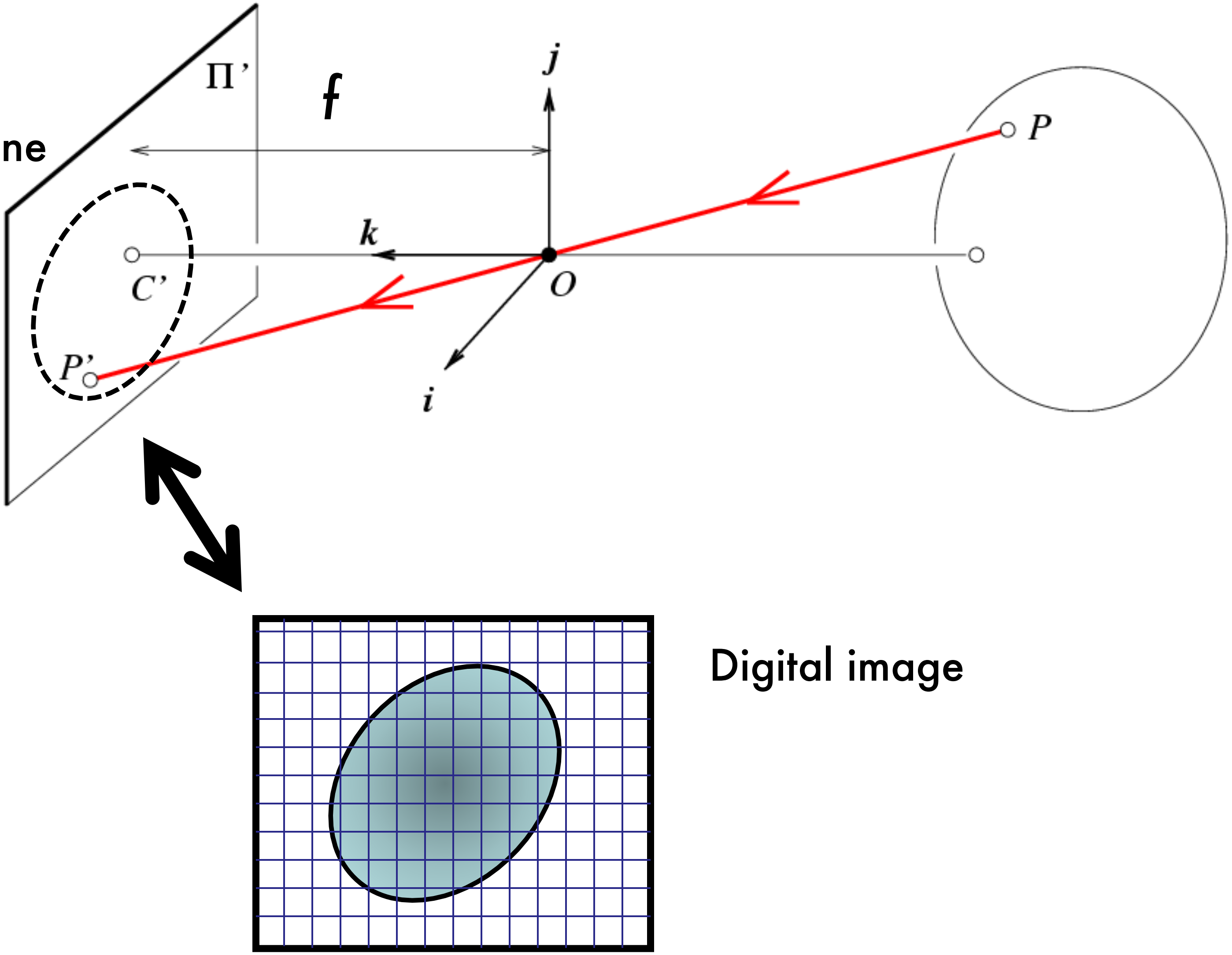
$$\begin{cases} x' = f \dfrac{x}{z} \\ y' = f \dfrac{y}{z} \end{cases}$$

$$\Re^3 \xrightarrow{E} \Re^2$$

[Eq. 1]

f = focal length
o = center of the camera

From retina plane to images

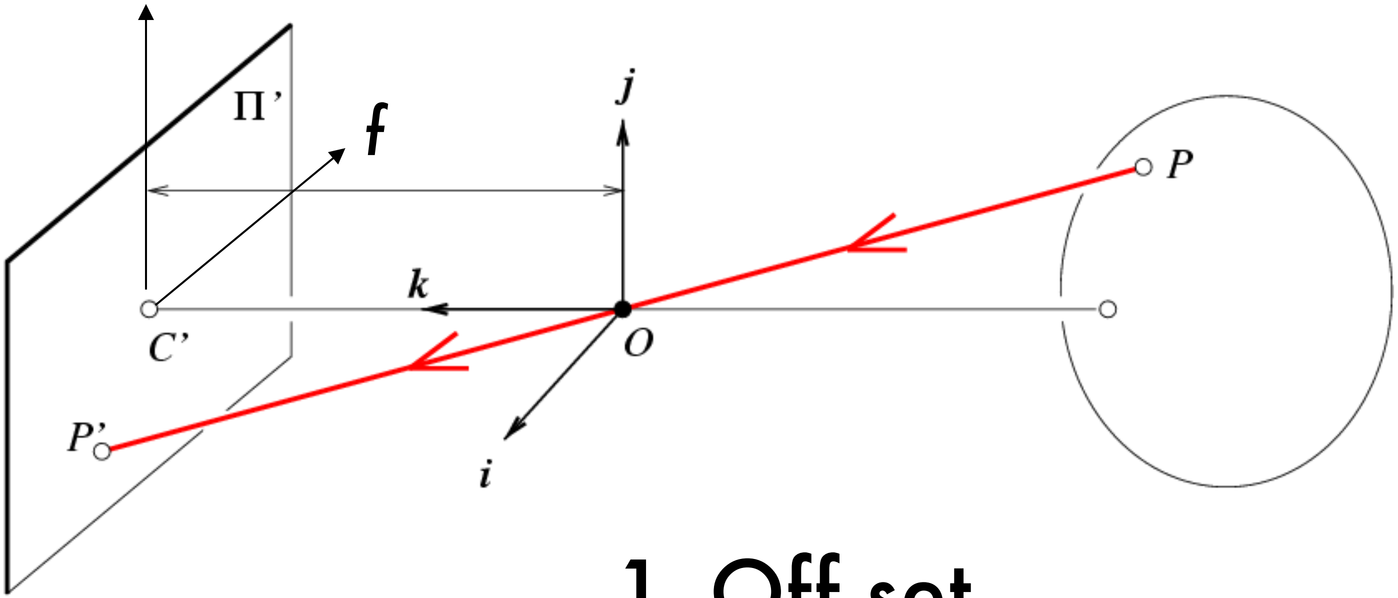


Retina plane

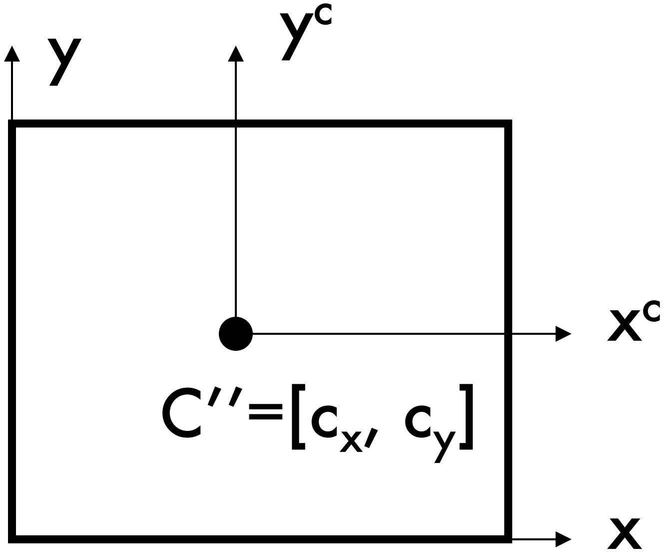
Digital image

Pixels, bottom-left coordinate systems

Coordinate systems



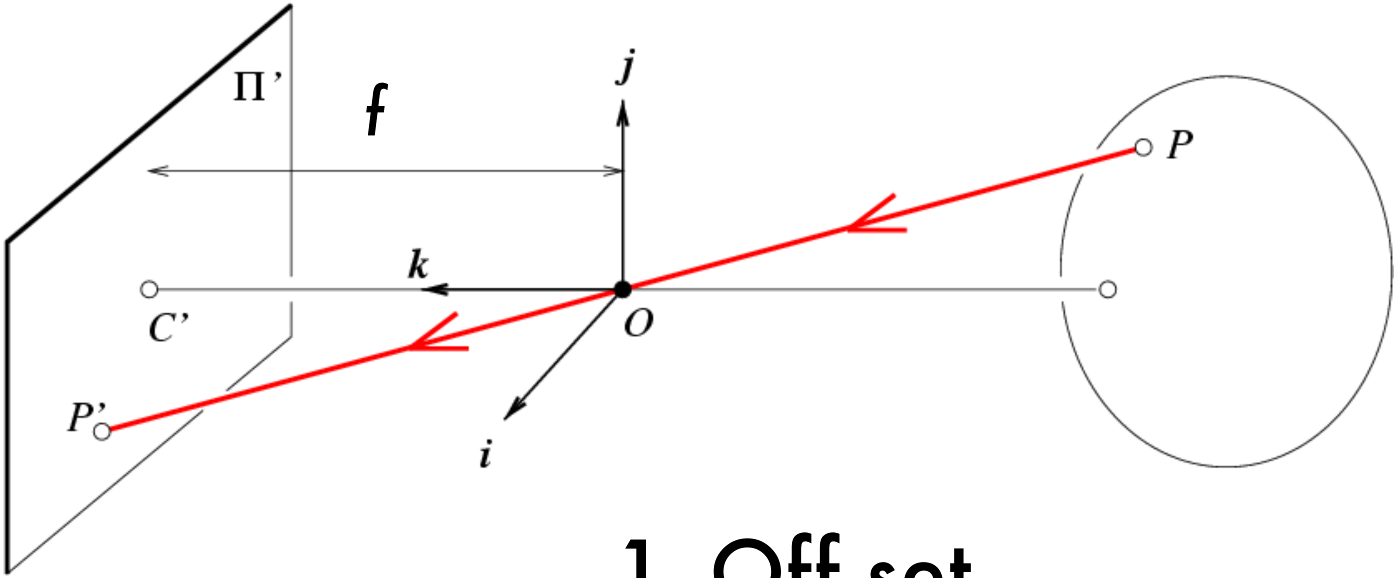
1. Off set



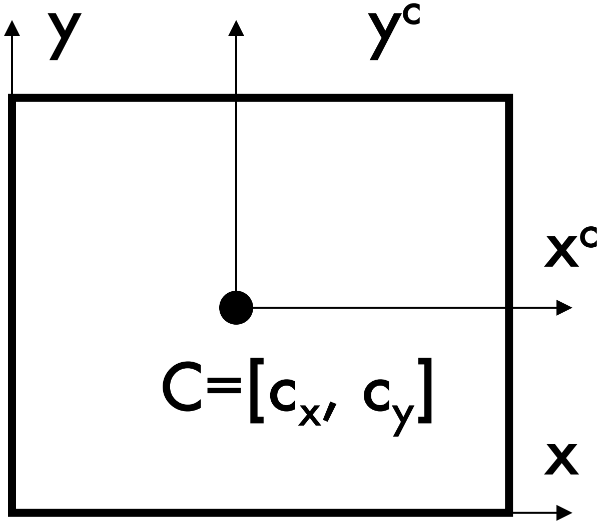
$$(x, y, z) \rightarrow (f\,\frac{x}{z} + c_x,\; f\,\frac{y}{z} + c_y)$$

[Eq. 5]

Converting to pixels



1. Off set
2. From metric to pixels



$$(x, y, z) \rightarrow \left(\underset{\alpha}{\boxed{f\,k}}\frac{x}{z}+c_x,\ \underset{\beta}{\boxed{f\,l}}\frac{y}{z}+c_y\right)$$

[Eq. 6]

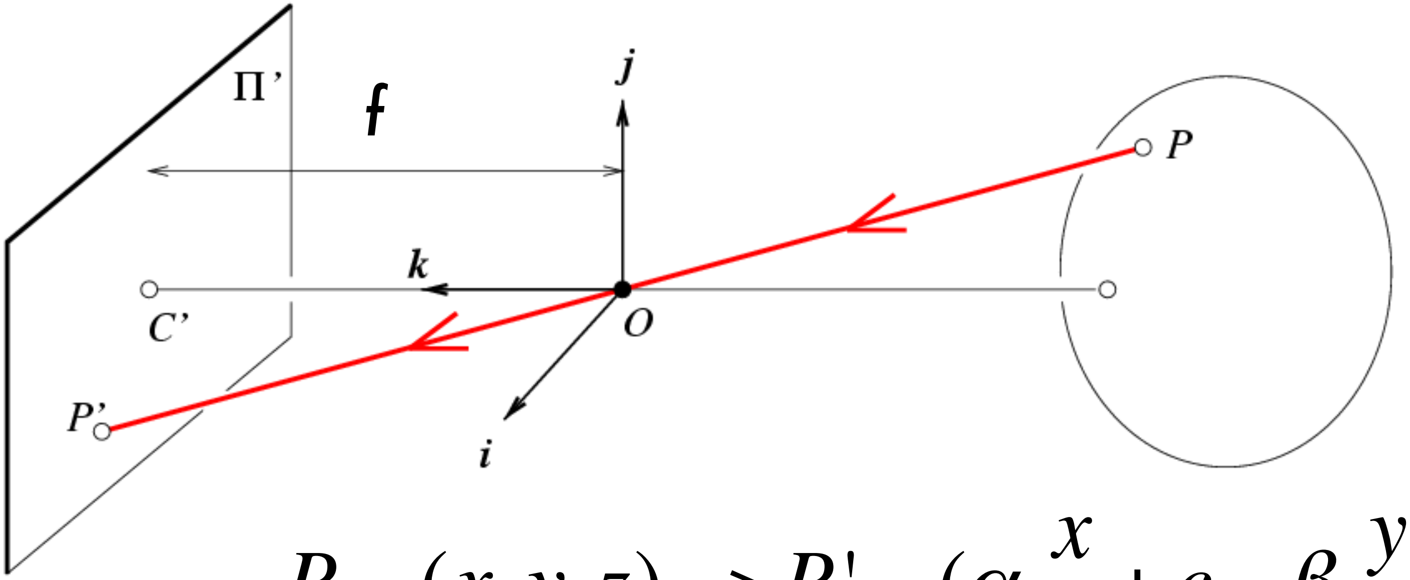
Units: k,l : pixel/m

f : m

Non-square pixels

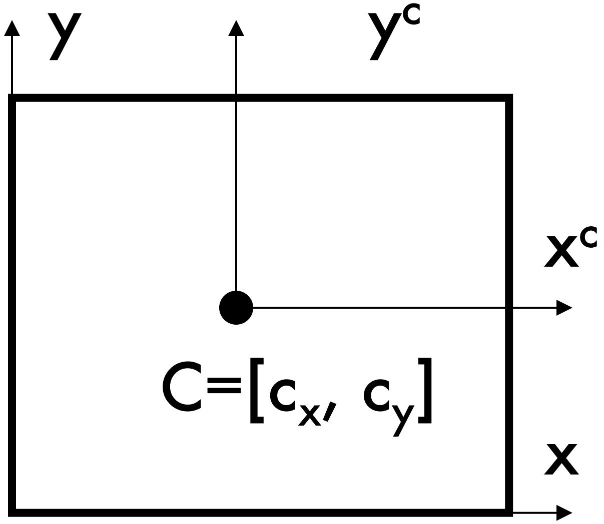
α, β : pixel

Is this projective transformation linear?



$$P=(x,y,z)\rightarrow P'=(\alpha\frac{x}{z}+c_x,\beta\frac{y}{z}+c_y)$$

[Eq. 7]



- Is this a linear transformation?

  No – division by z is nonlinear

- Can we express it in a matrix form?

Homogeneous coordinates

E→H

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

- Converting back *from* homogeneous coordinates

H→E

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Projective transformation in the homogenous coordinate system

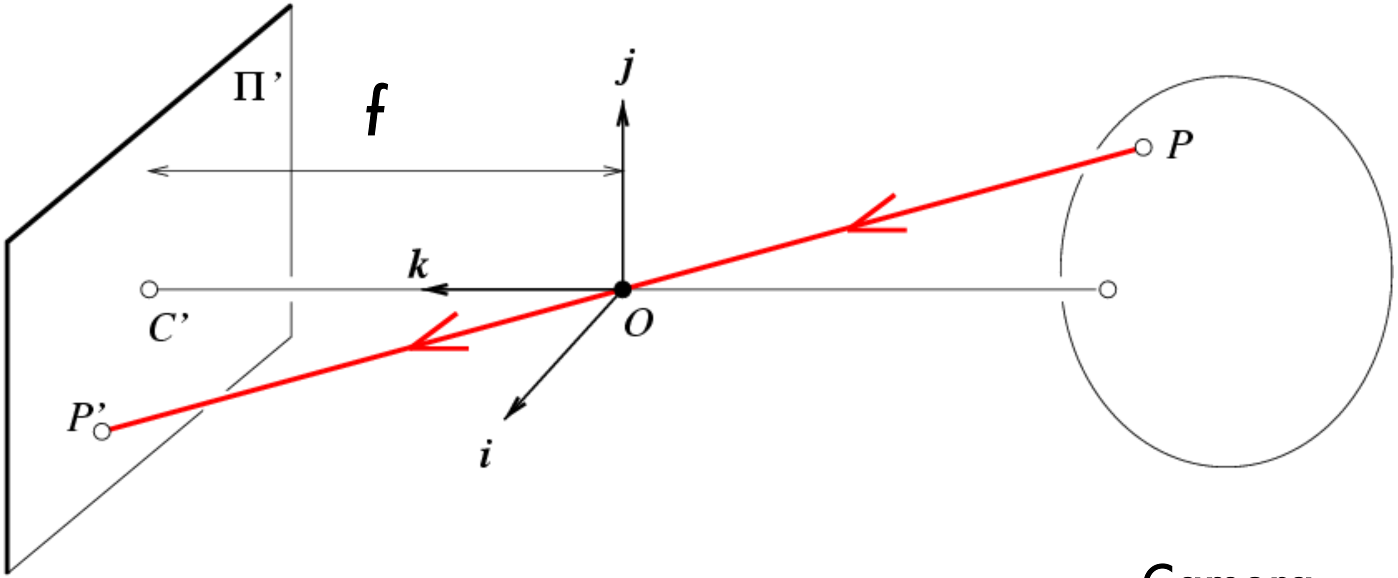
$$P_h' = \begin{bmatrix} \alpha\, x + c_x z \\ \beta\, y + c_y z \\ z \end{bmatrix} = \boxed{\begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \boxed{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}} \text{ — } P_h$$

[Eq.8]

$$M = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\overbrace{P_h'}^{\text{Homogenous}} \rightarrow \overbrace{P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)}^{\text{Euclidian}}$$

The Camera Matrix



Camera matrix K

[Eq.9]

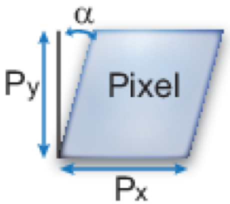
$$P' = M\,P$$

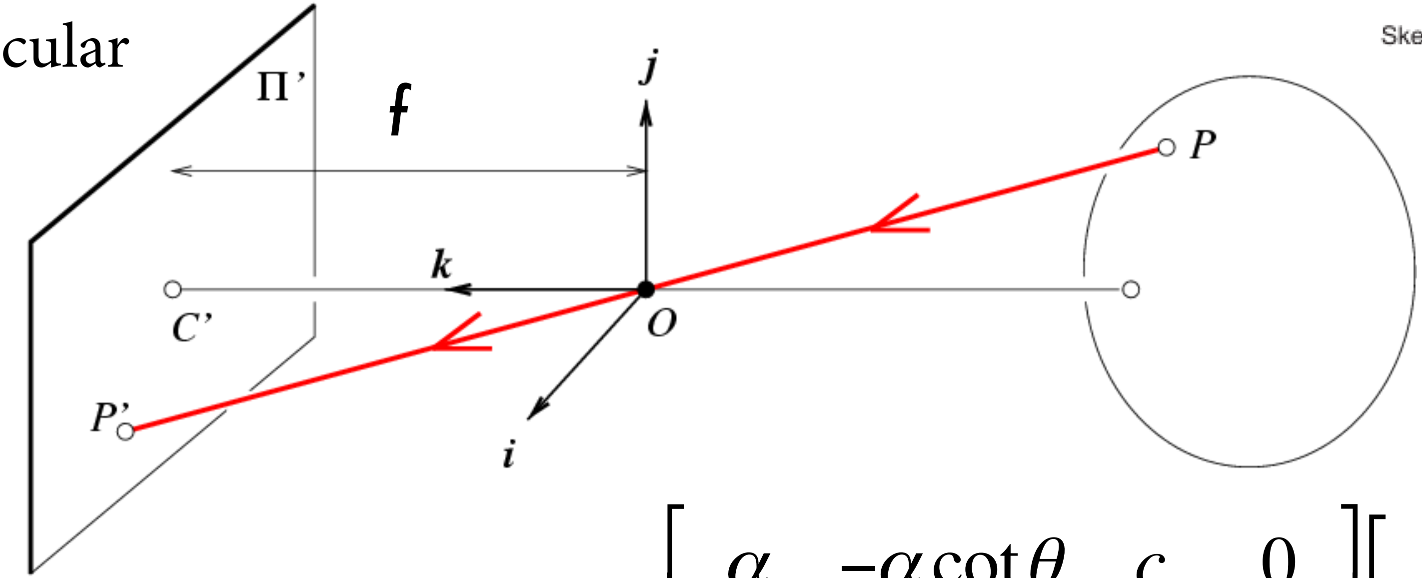
$$= K\begin{bmatrix} I & 0 \end{bmatrix} P$$

$$P' = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

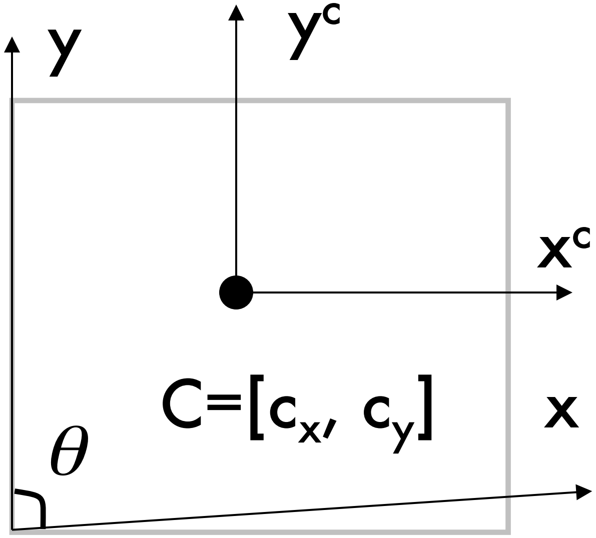
Camera Skewness

image
axis not
perpendicular





$$P' = \begin{bmatrix} \alpha & -\alpha \cot\theta & c_x & 0 \\ 0 & \dfrac{\beta}{\sin\theta} & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



How many degree does K have?
5 degrees of freedom!

Canonical Projective Transformation

[Eq.10]

$$P' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

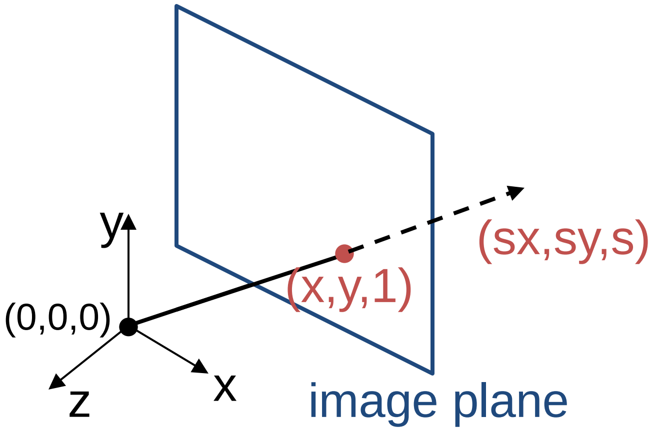
$$P' = M\,P$$

$$\Re^4 \xrightarrow{H} \Re^3$$

$$P_i' = \begin{bmatrix} \frac{x}{z} \\ \frac{y}{z} \end{bmatrix}$$

The projective plane

- Why do we need homogeneous coordinates?
  - represent points at infinity, homographies, perspective projection, multi-view relationships
- What is the geometric intuition?
  - a point in the image is a *ray* in projective space

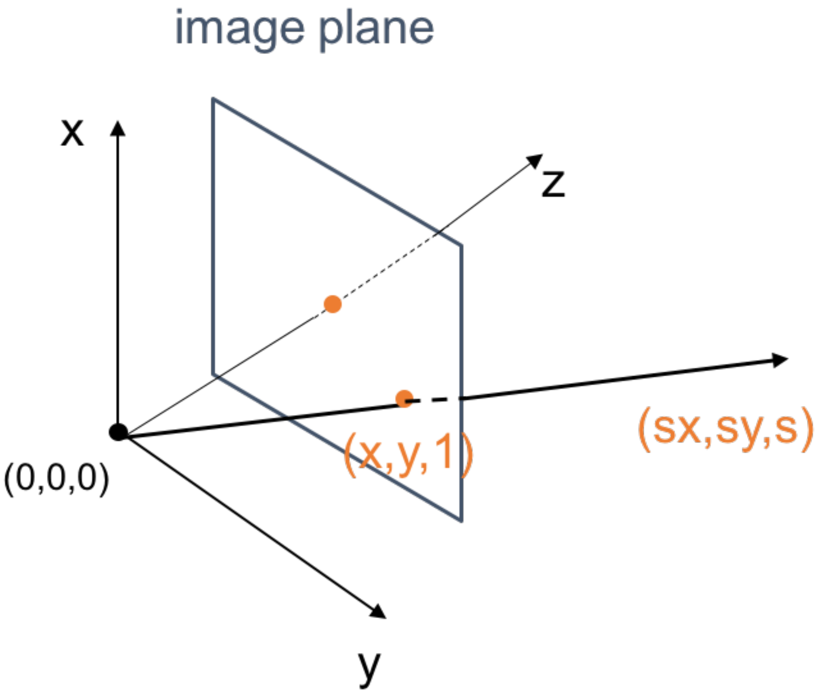


- Each *point* (x,y) on the plane is represented by a *ray* (sx,sy,s)
  - all points on the ray are equivalent: (x, y, 1) ≅ (sx, sy, s)

Point

Homogeneous coordinates

- – represent coordinates in 2 dimensions with a 3-vector



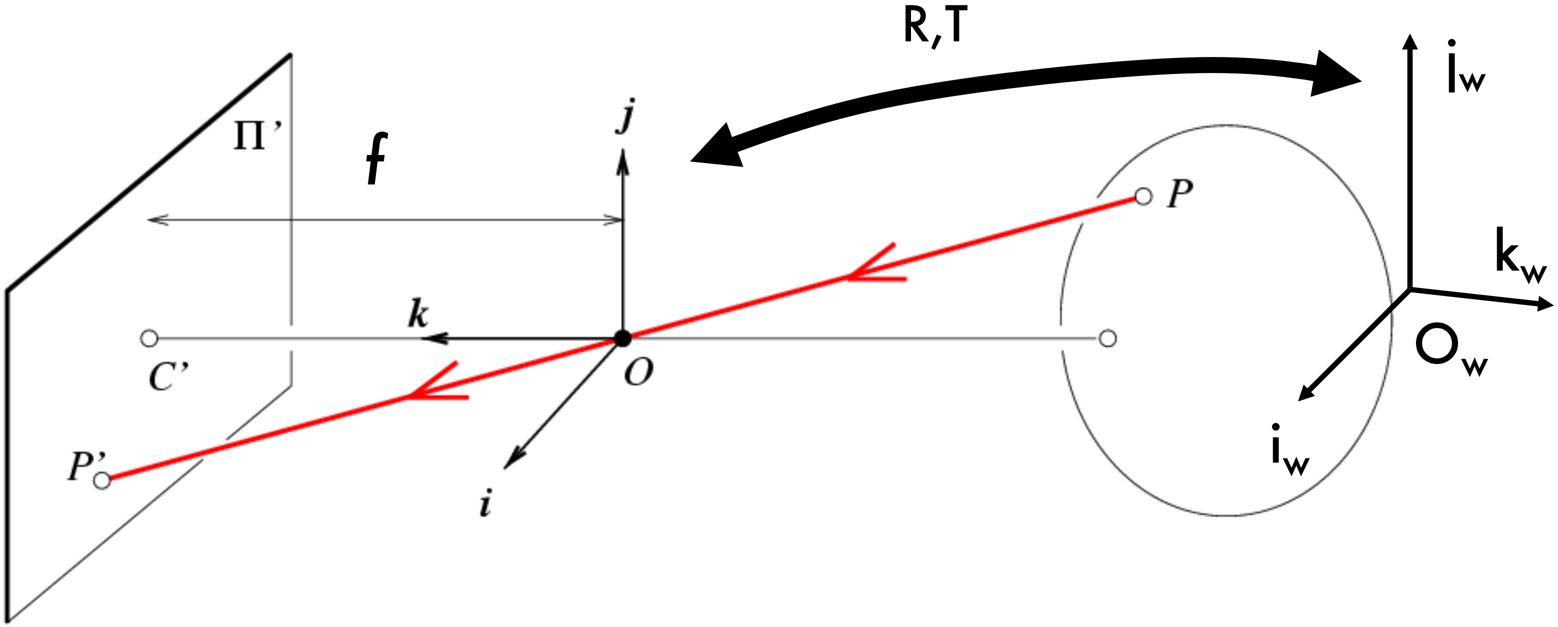
$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Lecture 2

Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
  - Intrinsic
  - Extrinsic
- Other camera models

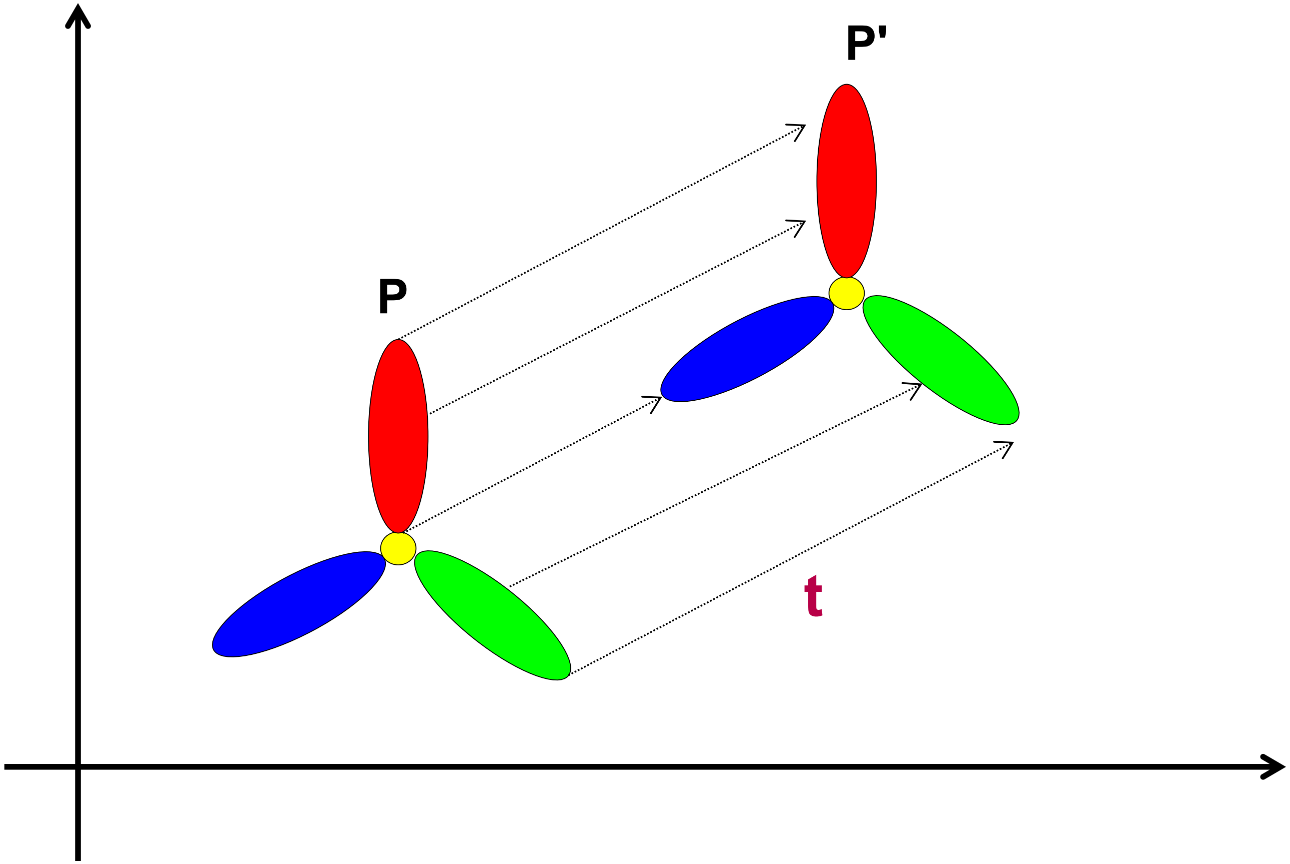
World reference system



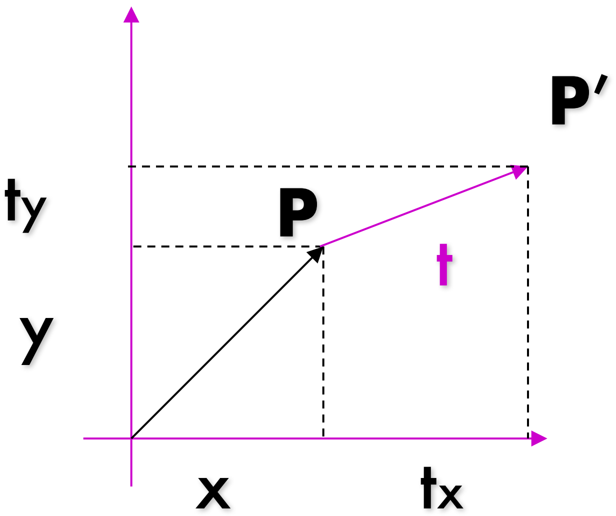
- The mapping so far is defined within the camera reference system
- What if an object is represented in the world reference system?
- Need to introduce an additional mapping from world ref system to camera ref system

Please refer to CA session
on transformations for
more details

2D Translation



2D Translation Equation

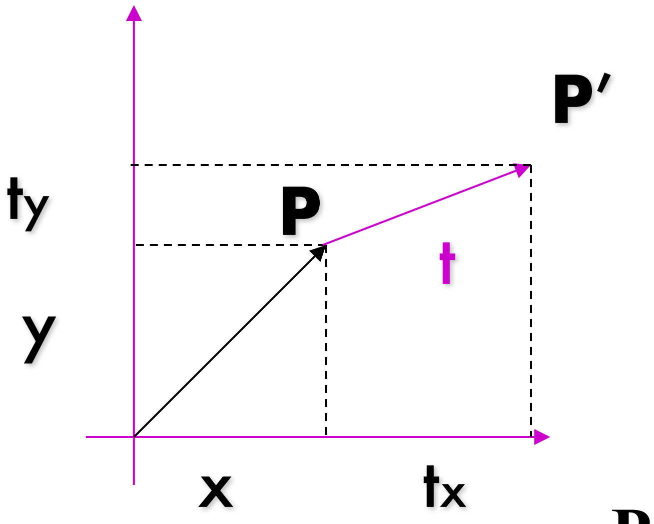


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (\mathrm{x} + \mathrm{t}_{\mathrm{x}}, \mathrm{y} + \mathrm{t}_{\mathrm{y}})$$

2D Translation using Homogeneous Coordinates

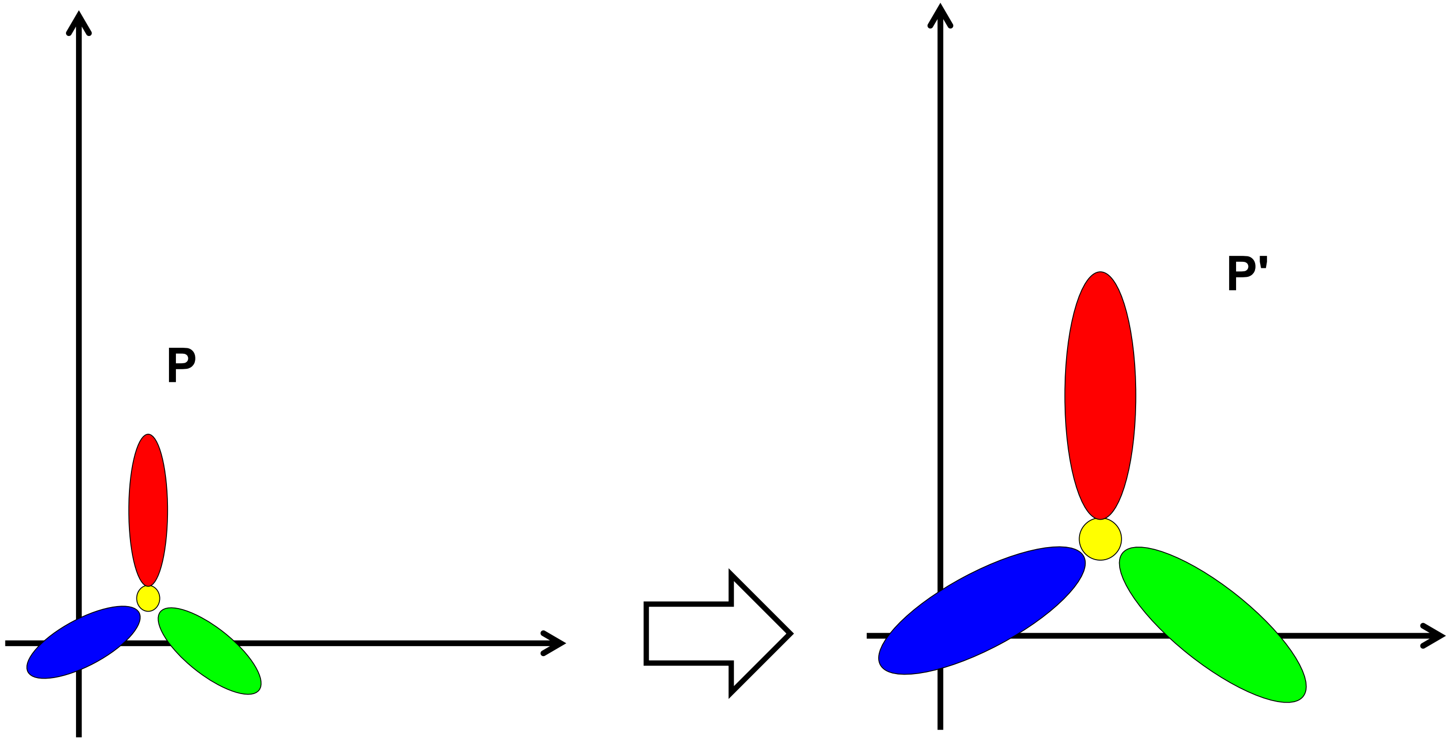


$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

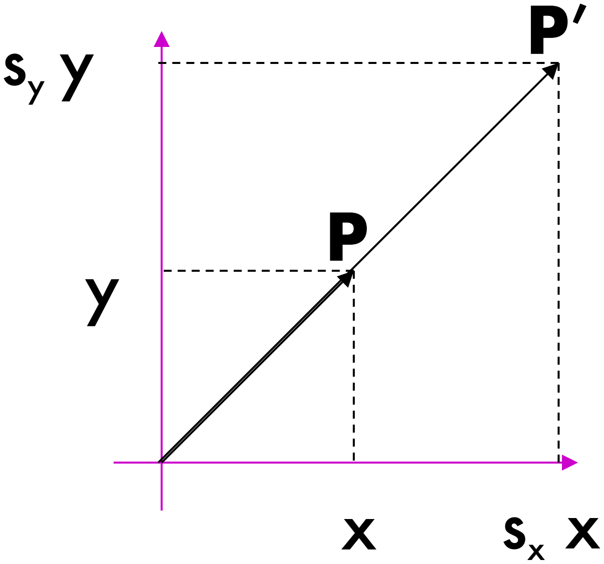
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{T} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling



Scaling Equation

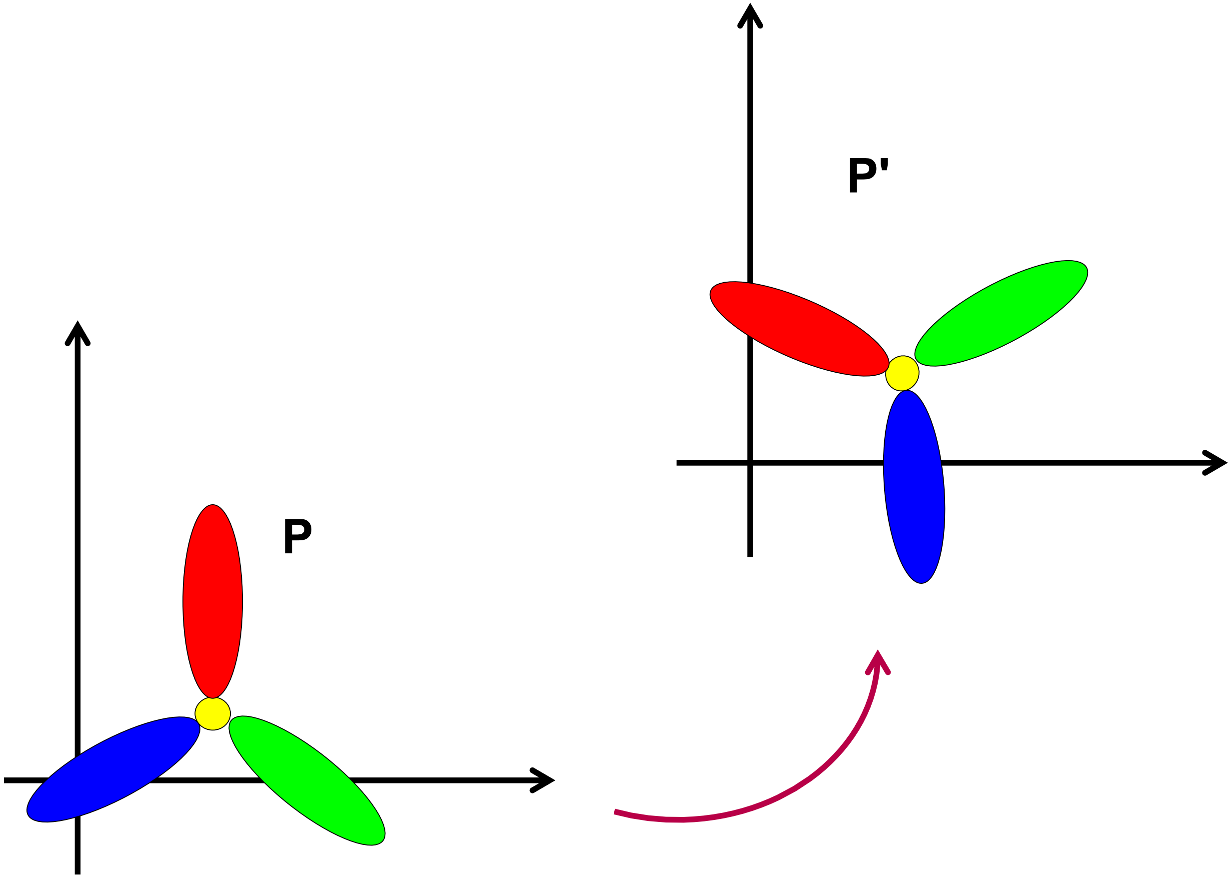


$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P}' = (\mathrm{s}_\mathrm{x}\mathrm{x}, \mathrm{s}_\mathrm{y}\mathrm{y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

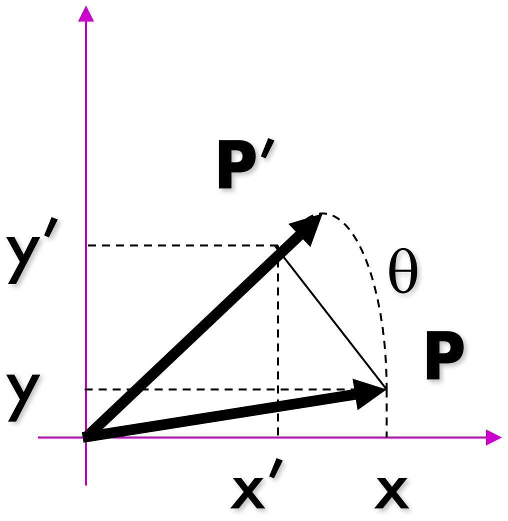
$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{S} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

- Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\, x - \sin\theta\, y$$

$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\ \mathbf{P}$$

How many degrees of freedom? 1

$$\mathbf{P'} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale + Rotation + Translation

$$\mathbf{P}' \rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left[\boxed{\begin{matrix} \mathbf{R}\,\mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{matrix}} \right] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x = s_y$, this is a similarity transformation

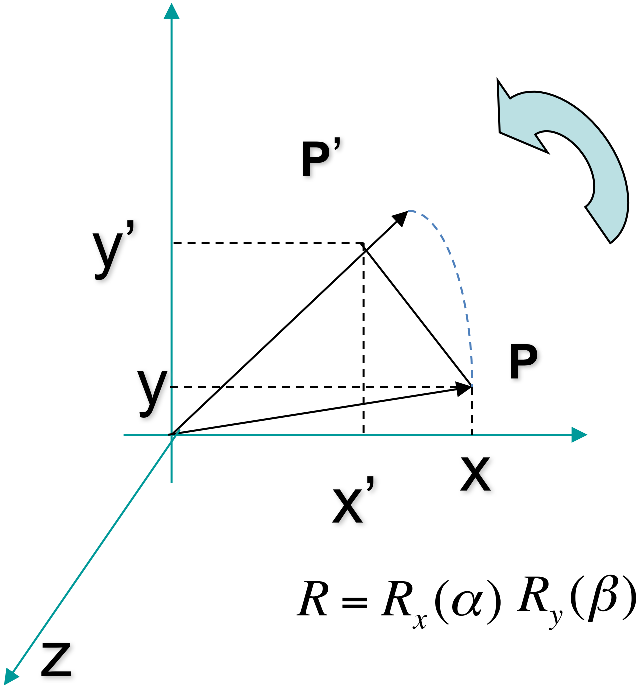
3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

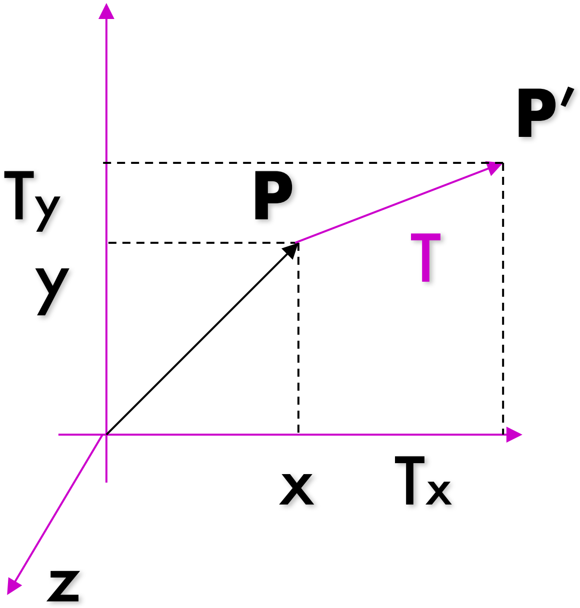


$$R = R_x(\alpha)\ R_y(\beta)\ R_z(\gamma)$$

$$P' \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}_{4\times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A rotation matrix in 3D has 3 degrees of freedom

3D Translation of Points



$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix}_{4\times4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

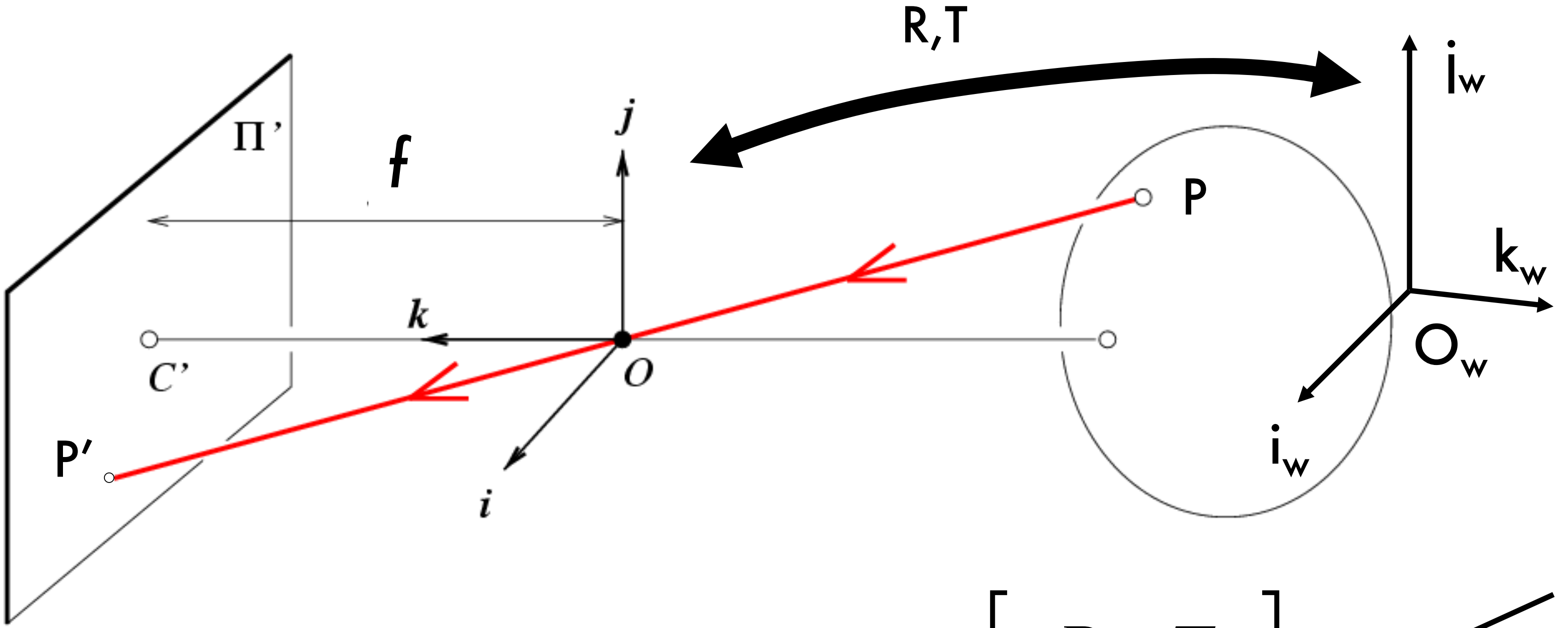
A translation vector in 3D has 3 degrees of freedom

3D Translation and Rotation

$$R = R_x(\alpha)\ R_y(\beta)\ R_z(\gamma) \qquad T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

World reference system



In 4D homogeneous coordinates: $P = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} P_w$

$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

Internal parameters

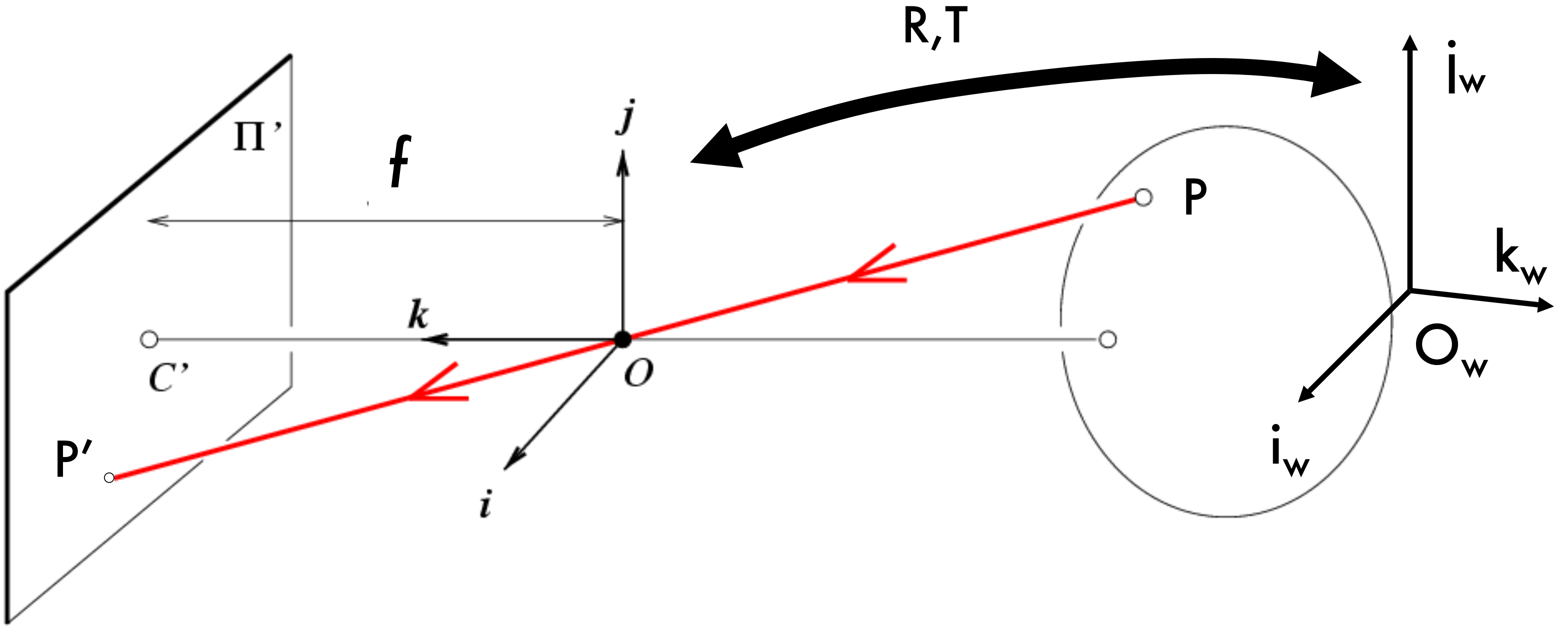
External parameters

$$P' = K\begin{bmatrix} I & 0 \end{bmatrix} P = K\begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} P_w = \boxed{\boxed{K}\begin{bmatrix} \boxed{R \quad T} \end{bmatrix}} P_w$$

M

[Eq.11]

The projective transformation

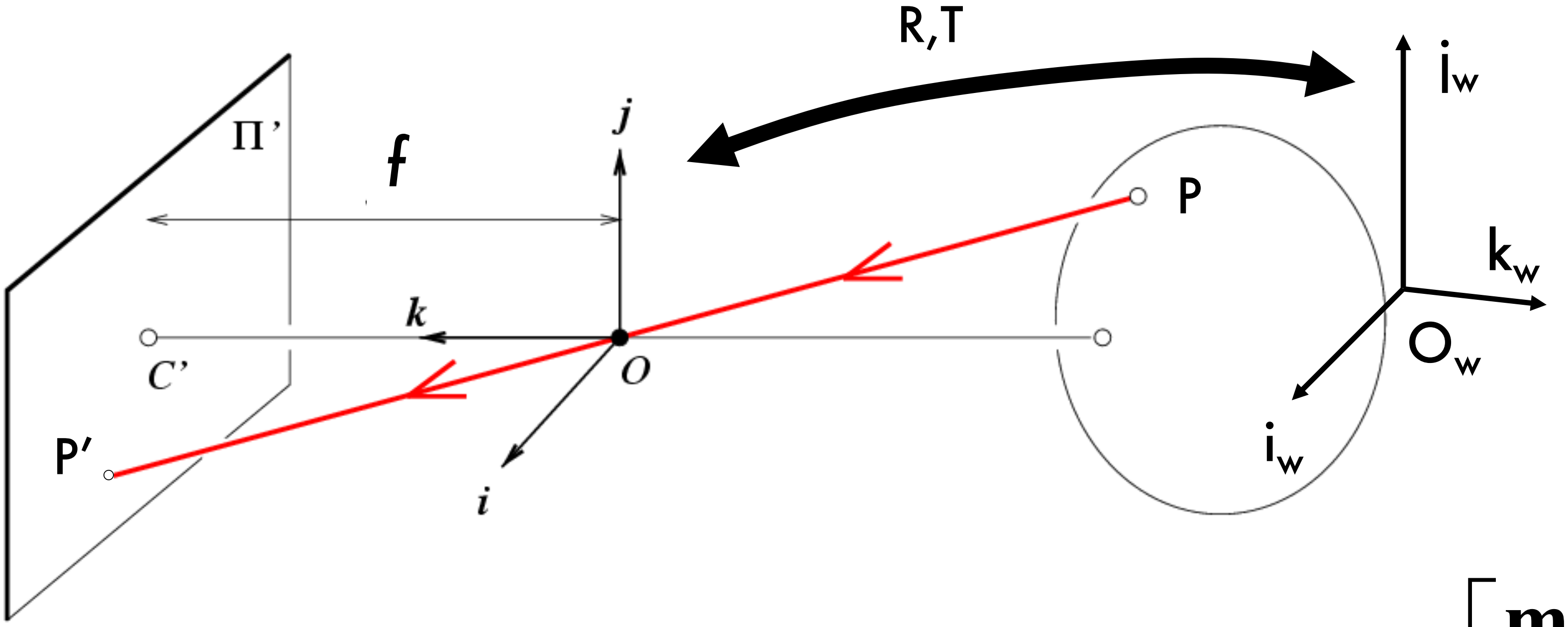


$$P'_{3\times1} = M_{3x4}\, P_w \;= K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4} P_{w\,4\times1}$$

How many degrees of freedom?

5 + 3 + 3 =11!

The projective transformation



$$P'_{3\times 1} = M\ P_w \quad = K_{3\times 3}\begin{bmatrix} R & T \end{bmatrix}_{3\times 4} P_{w\,4\times 1} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

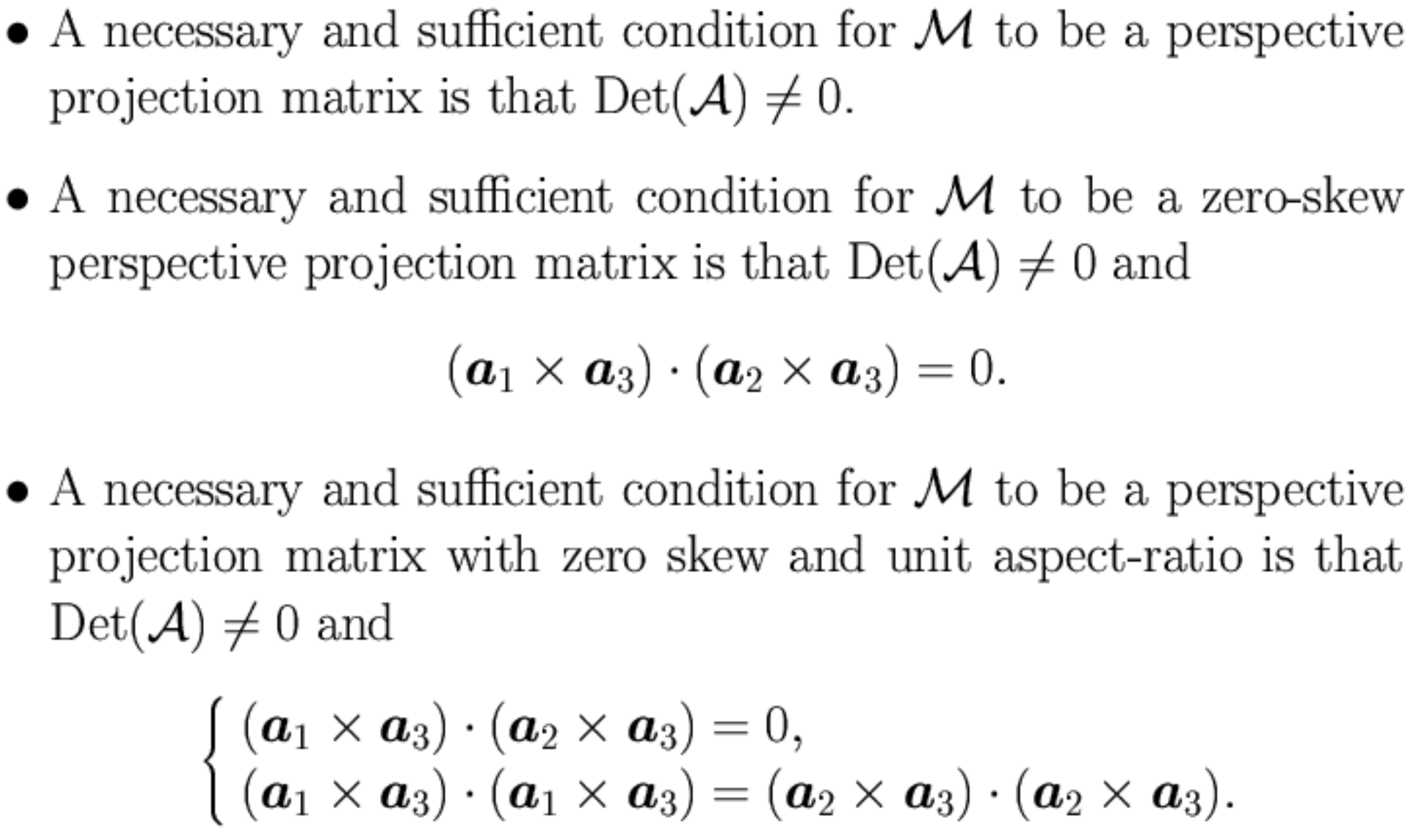
$$= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_{\mathrm{W}} = \begin{bmatrix} \mathbf{m}_1 P_{\mathrm{W}} \\ \mathbf{m}_2 P_{\mathrm{W}} \\ \mathbf{m}_3 P_{\mathrm{W}} \end{bmatrix} \overset{\mathbf{E}}{\rightarrow} \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right)$$

[Eq.12]

Theorem (Faugeras, 1993)

$$M = K\begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} KR & KT \end{bmatrix} = \begin{bmatrix} A & b \end{bmatrix} \qquad A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

[Eq.13]



- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

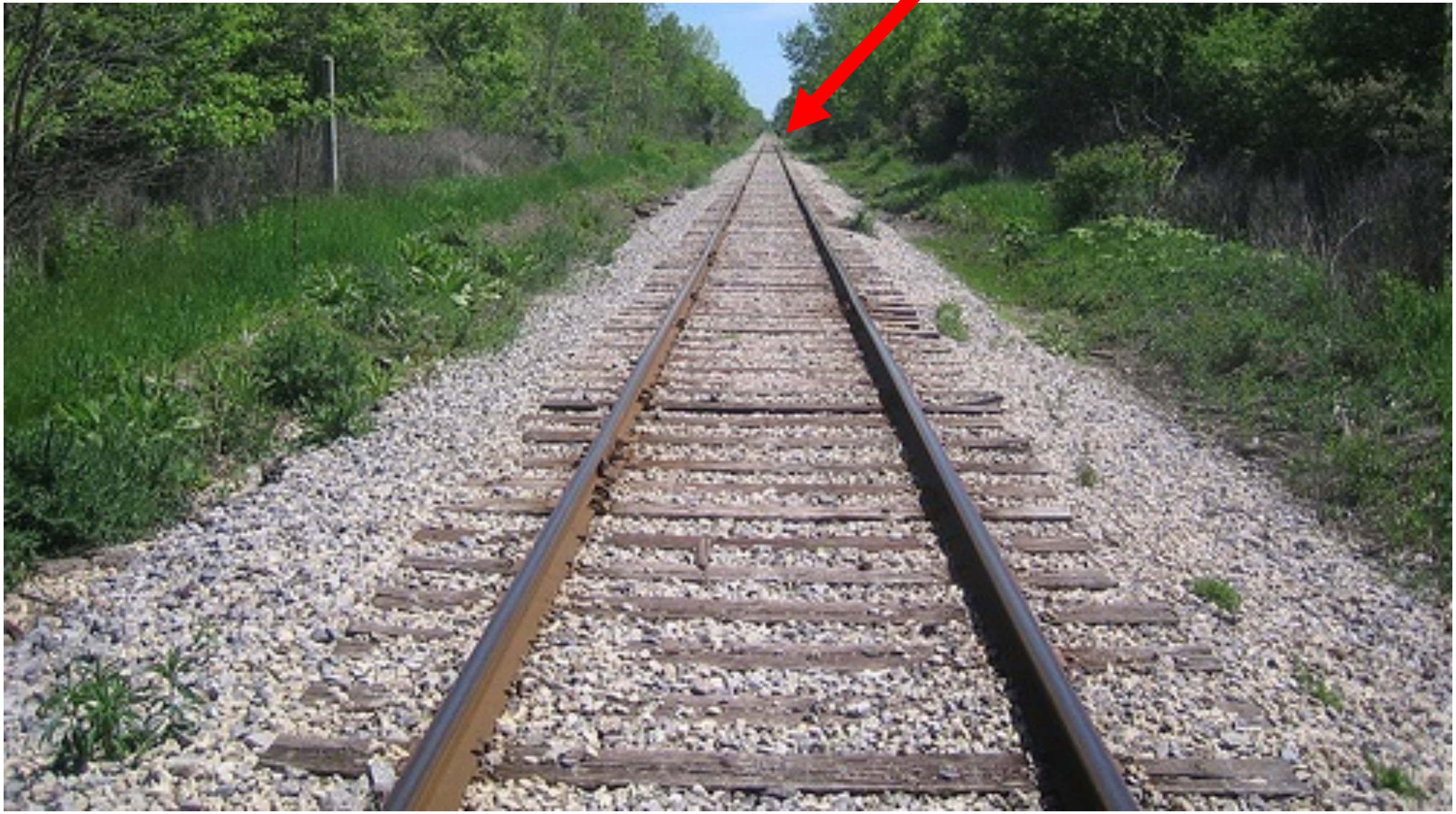
$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

Properties of projective transformations

- Points project to points
- Lines project to lines
- Distant objects look smaller

Properties of Projection

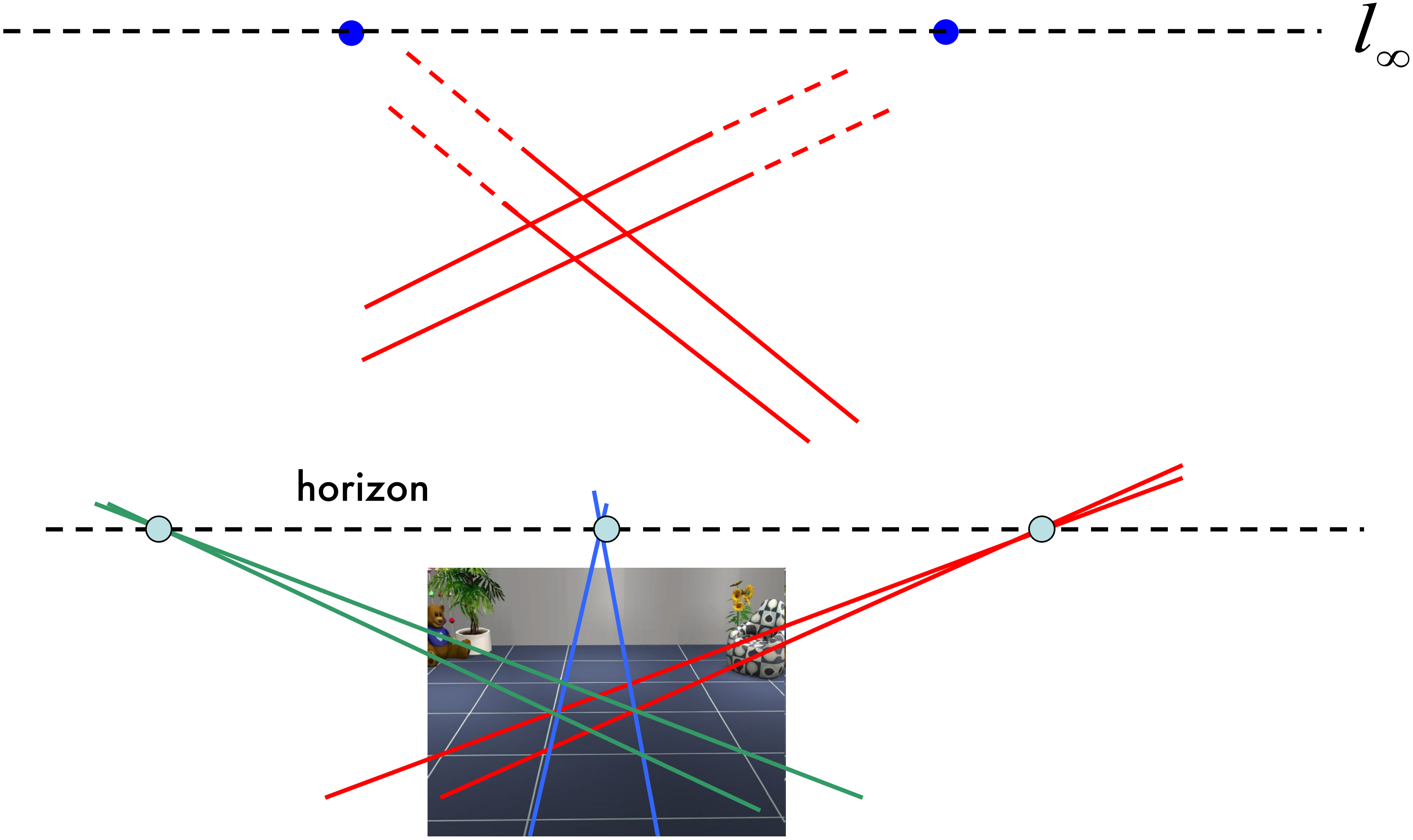
- Angles are not preserved
- Parallel lines meet!



Parallel lines in the world
intersect in the image at a
“vanishing point”

Horizon line (vanishing line)

Horizon line (vanishing line)



Next lecture

- How to calibrate a camera?