EE40
Lecture 11
Josh Hug

7/19/2010

Logistical Things

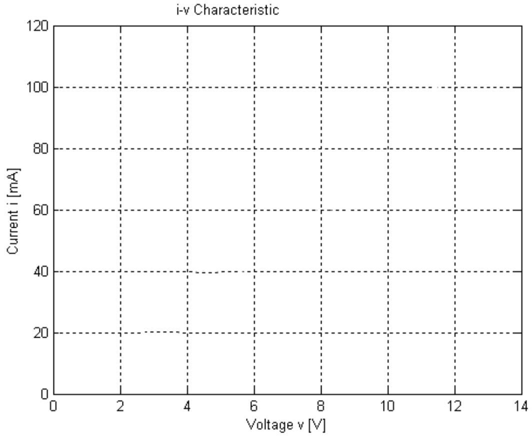
- Lab 4 tomorrow
- Lab 5 (active filter lab) on Wednesday
  - Prototype for future lab for EE40
  - Prelab is very short, sorry.
  - Please give us our feedback
  - Google docs for labs and general class comments now available (link shared via email)
  - Bring a music player if you have one (if not, you can use the signal generator in the lab)
- HW5 due tomorrow at 2PM
- HW6 due Friday at 5PM (also short)
- Midterm next Wednesday 7/28
  - Focus is heavily on HW4, 5, 6, and Labs P1, 4, 5
  - Will reuse concepts from HW 1,2,3

Logistics

- No lunch today
- Slightly shorter lecture today
- Midterm regrade requests due today
- Office hours Cory 240 2:30-4PM or so

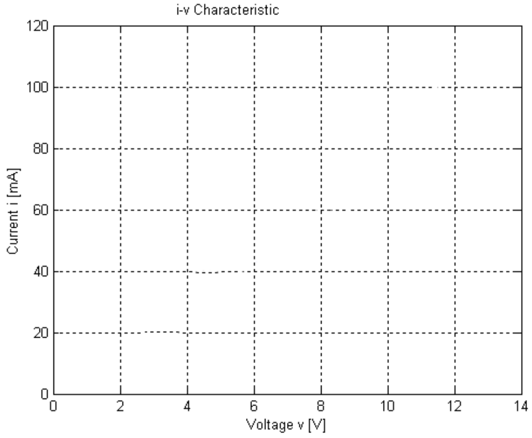
iClicker Question #1

- Consider a capacitor with a capacitance of $1nF$. Total resistance of circuit is small $\sim 1\Omega$.
- If we **compile an I-V table in lab** with a voltage source and multimeter by:
  - Applying a set of test voltages
  - Measuring current through the capacitor for each source
- What I-V characteristic will we get?
  - A. Horizontal line at I=0
  - B. Vertical line at I=0
  - C. Line of slope 1/RC
  - D. Horizontal line at I=V/1Ω
  - E. Something else



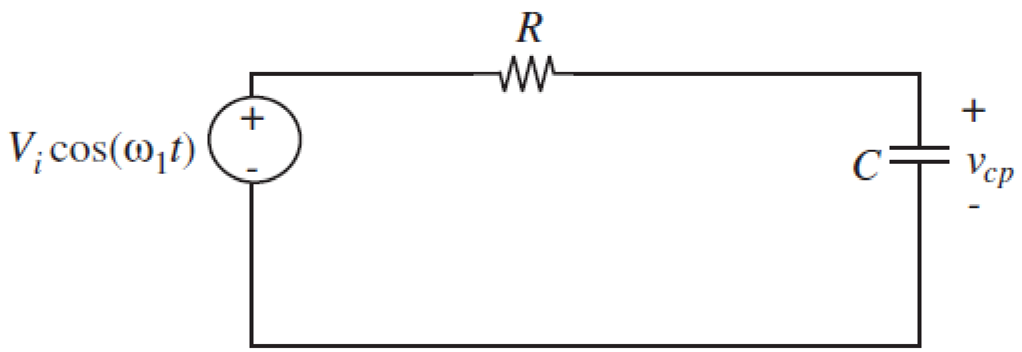
iClicker Question #2

- Consider an inductor with an inductance of 1μH. Total resistance of circuit is small ~1Ω.
- If we **compile an I-V table in lab** with a voltage source and multimeter by:
  - – Applying a set of test voltages
  - – Measuring current through the inductor for each source
- What I-V characteristic will we get?
  - – A. Horizontal line at I=0
  - – B. Vertical line at I=0
  - – C. Line of slope R/L
  - – D. Line at I=V/1Ω
  - – E. Something else



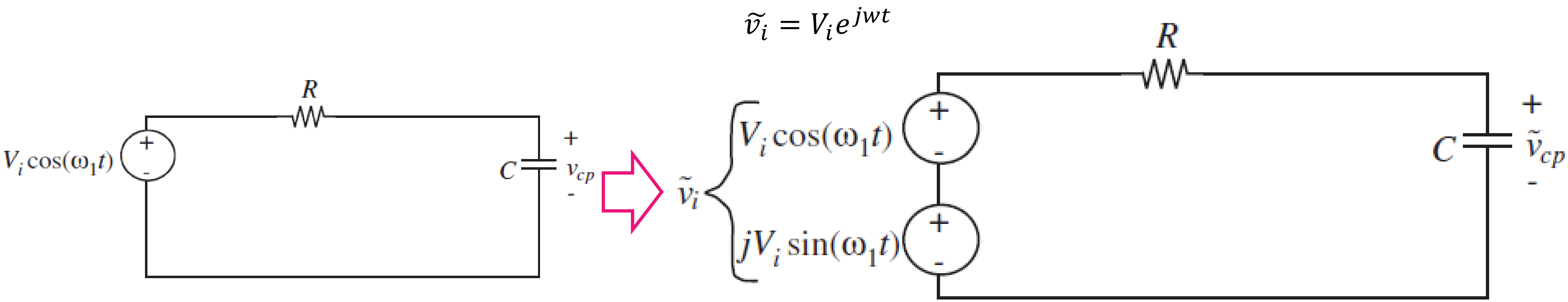
Easy Method for AC Circuits

- We want to find the voltage across the capacitor for the following circuit



- Homogenous solution is easy, since source is irrelevant
- Finding particular solution the usual way (plugging in a guess, finding coefficients that cancel) is painful

Easy Method for AC Circuits



Guess $v_{cp} = Acos(\omega_1 t + \phi)$

Plug into ODE

⇩

Solve for A and ϕ (hard)

⇩

$$v_{cp} = Acos(\omega_1 t + \phi)$$

Guess $\tilde{V}_{cp} = k_1 e^{j\omega t}$

⇩

Plug into ODE

⇩

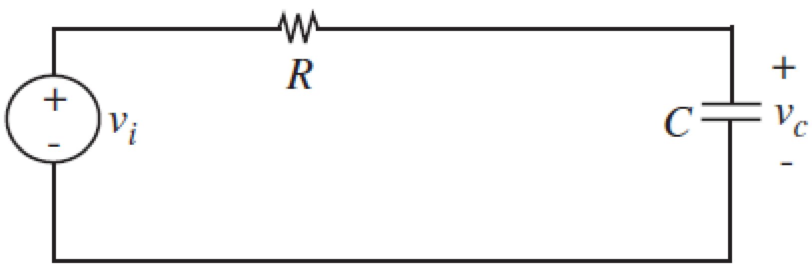
Divide by $e^{j\omega t}$. Now t is gone

Solve for k_1 (easy)

⇩

$$v_{cp} = Real[k_1 e^{j\omega t}]$$

Memory Circuits with Exponential Source



$$v_I = V_i e^{jwt} \quad t > 0$$

$$V_O' = -\frac{V_O}{RC} + V_i \frac{e^{j\omega t}}{RC}$$

- Homogeneous solution is just $Ae^{-t/RC}$
- Pick particular solution $V_{O,P} = k_1 e^{jwt}$, plug in:

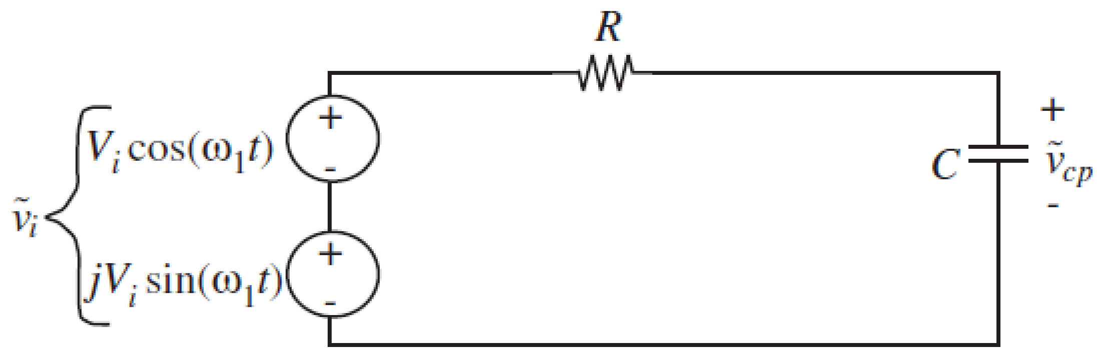
$$k_1 j\omega e^{j\omega t} = -k_1 \frac{e^{j\omega t}}{RC} + V_i \frac{e^{j\omega t}}{RC}$$

- Divide by e^{jwt}, and solve for k_1

$$k_1 = V_i \frac{1}{1 + j\omega RC} \qquad V_{O,P}(t) = V_i \frac{1}{1 + j\omega RC} e^{jwt}$$

Real part of $V_{O,P}$ gives solution for cosine source

Inverse Superposition



- Superposition tells us that our output $V_{O,P}(t)$ will just be the sum of the effect of these two sources

$$V_{O,P}(t) = V_i \frac{1}{1 + j\omega RC} e^{jwt}$$

- Luckily for us, all complex numbers are the sum of their real and imaginary parts $\mathrm{x} = a + jb$
- Just find real part and we're done!

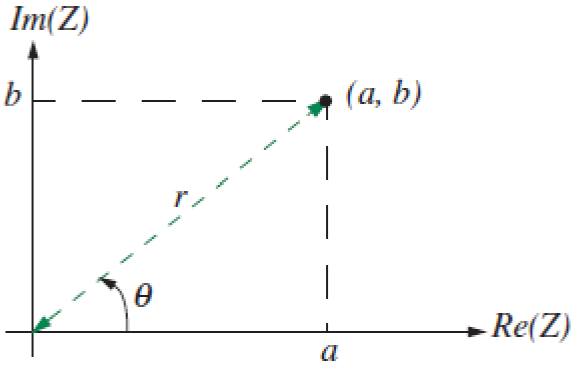
Real Part of Expression

- Finding the real part of the expression is easy, it just involves some old school math that you've probably forgotten (HW5 has complex number exercises)

$$V_{O,P}(t) = \frac{1}{1 + j\omega RC} V_i e^{jwt}$$

- Key thing to remember is that complex numbers have two representations
  - Rectangular form: $a + jb$
  - Polar form: $re^{j\theta}$

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$



Real Part of Expression

- What we have is basically the product of two complex numbers
- Let's convert the left one to polar form

$$V_{O,P}(t) = \frac{1}{1 + j\omega RC} V_i e^{jwt}$$

  - Rectangular form: $a + jb$
  - Polar form: $re^{j\theta}$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$V_{O,P}(t) = \frac{1}{Re^{j\phi}} V_i e^{jwt} = V_i \frac{1}{\sqrt{1 + (wRC)^2}} e^{\phi j} e^{jwt}$$

$$\phi = \arctan(\omega RC)$$

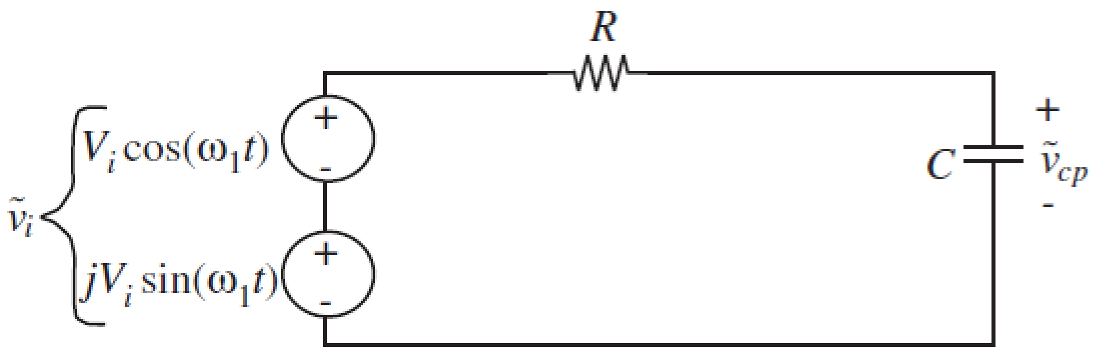
Real Part of Expression

$$V_i \frac{1}{\sqrt{1+(wRC)^2}} e^{j\phi} e^{jwt}$$

$$V_i \frac{1}{\sqrt{1+(wRC)^2}} e^{j(\phi+\omega t)}$$

$$V_i \frac{1}{\sqrt{1+(wRC)^2}} (\cos(\omega t+\phi)+jsin(\omega t+\phi))$$

Real Part of Expression



- Superposition tells us that our output $V_{O,P}(t)$ will just be the sum of the effect of these two sources

$$V_{O,P}(t) = \frac{V_i}{\sqrt{1+(wRC)^2}}(\cos(\omega t+\phi)+jsin(\omega t+\phi))$$

- Thus, particular solution (forced response) of original cosine source is just the real part

$$V_{O,P}(t) = \frac{V_i}{(1+\omega RC)^2}\cos(\omega t+\phi) \qquad \phi = \arctan(\omega RC)$$

Easy Method for AC Circuits

Just as actually writing the ODE isn't necessary for DC sources, we can avoid the ODE again in AC circuits:

Impedance Analysis

Write ODE

⇩

Guess $\tilde{V}_{cp} = k_1 e^{j\omega t}$

⇩

Plug into ODE

⇩

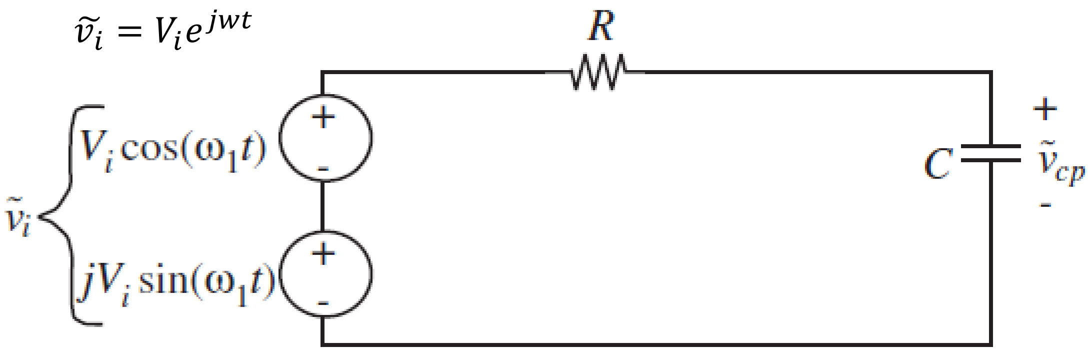
Divide by $e^{j\omega t}$. Now t is gone

Solve for k_1 (easy)

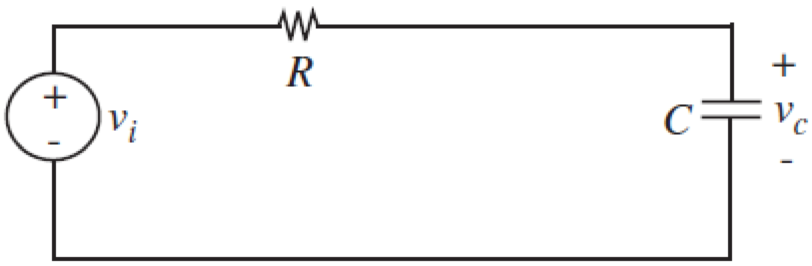
⇩

$$v_{cp} = Real[k_1 e^{j\omega t}]$$

$\tilde{v}_i = V_i e^{jwt}$



Impedance



$$v_I = V_i e^{j\omega t} \quad t > 0$$

For a complex exponential source:

$$V_{C,P}(t) = \frac{1}{1 + j\omega RC} v_I(t)$$

Rewrite as:

$$V_{C,P}(t) = \frac{1/jwC}{1/jwC + R} v_I(t)$$

Let $Z_c = 1/jwC$

$$V_{C,P}(t) = \frac{Z_c}{Z_c + R} v_I(t)$$

Looks a lot like…

voltage divider

Real part gives solution for $v_I = V_i \cos(\omega t)$

Method of Impedance Analysis (without Phasors)

- Replace passive components with equivalent impedance, $Z_c = \frac{1}{j\omega C}$, $Z_L = j\omega L$, $Z_R = R$
- Replace all sources with complex exponentials
  - e.g. $v(t) = A\cos(\omega t + \theta) \Longrightarrow \tilde{v}(t) = Ae^{j(wt+\theta)}$
- Solve using Ohm's Law of Impedances for complex exponential sources
  - $\tilde{v}(t) = \tilde{\imath}(t)Z$
  - Just like normal node voltage, but with complex numbers
  - Real part of node voltage $\tilde{V}_a(t)$ gives true output $V_a(t)$

Lugging these complex exponential functions is algebraically annoying

Phasors (not in the book!)

- Definition: A phasor is a complex number which represents a sinusoid
- $f(t) = Acos(\omega t + \theta)$
- Three parameters
  - A: Magnitude
  - ω: Frequency
  - θ: Phase
- The phasor representation of the sinusoid above is $Ae^{j\theta}$
- In shorthand we write phasor as $A\angle\theta$

Phasors

- If we have a voltage $V(t) = A cos(\omega t + \theta)$
- The phasor version of the voltage is $\hat{V} = A\angle\theta$
- If we have a phasor $\hat{I} = \alpha\angle\phi$, the time function this phasor represents is $i(t) = \alpha\cos(\omega t + \phi)$

Why are phasors useful?

- Sources that look like $Ae^{t(j\omega+\theta)}$ result in lots of $Ae^{t(jw+\theta)}$ terms in our algebra
- When you apply a sinusoidal source to a circuit, the amplitude and phase will vary across components, but it will always still be $\alpha e^{t(j\omega+\phi)}$
  - Important: ω doesn't change!
    - Otherwise we'd need REALLY complex numbers
- Thus, we'll just replace our sources with a complex number $A\angle\phi$ and just keep in mind that this number represents a function throughout

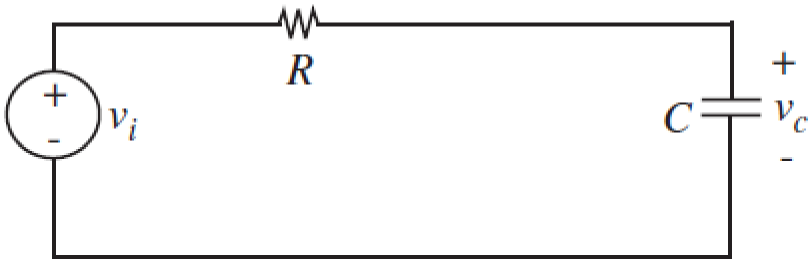
Why are phasors useful?

- We know that for complex exponential sources, we have that:
  - $\tilde{v}(t) = \tilde{i}(t)Z$
  - $real[\tilde{v}(t)] = real[\tilde{i}(t)Z]$
- Phasors are complex numbers $\hat{V}$ and $\hat{I}$ which represent cosine functions $v(t)$ and $i(t)$
- Cosine functions are just the real parts of complex exponentials
- Thus, in the world of phasors, we can just rewrite Ohm's Law of Impedances as:
  - $\hat{V}=\hat{I}Z$

Method of Impedance Analysis (with Phasors)

- Replace passive components with equivalent impedance, $Z_c = \frac{1}{j\omega C}, Z_L = j\omega L, Z_R = R$
- Replace all sources with phasor representation: e.g. $v(t) = A\cos(\omega t + \theta) \Longrightarrow \hat{V}(t) = A\angle\theta$
- Solve using Ohm's Law of Impedances:
  - $\hat{v} = \hat{\imath}Z$
  - Just like normal node voltage, but with complex numbers, attaining voltage phasors $\hat{V}_a, \hat{V}_b$, ...
  - Output $V_a(t)$ is just $|\hat{V}_a|\cos(\omega t + \angle\hat{V}_a)$
- Original sources are implicitly represented by phasors
  - Time is gone completely from our problem

Example



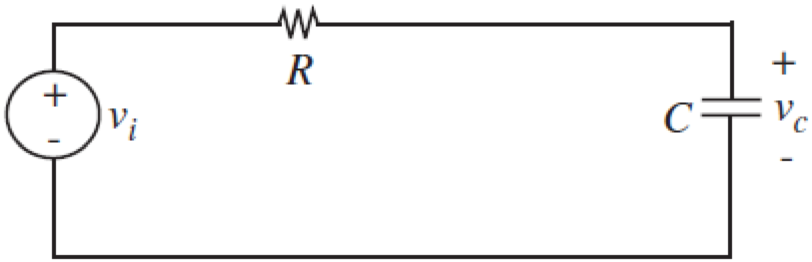
$$v_I = V_i cos(\omega t), \qquad t > 0$$

$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$
$$\omega = 100$$

Find $i(t)$ in steady state

- $Z_R = 10000,\ Z_C = \frac{1}{j\omega C} = -10000j$
- $\hat{V} = V_i\angle 0 = 5\angle 0$
- $Z_{eq} = 10000 - 10000j$
- $\hat{I} = \frac{5\angle 0}{10000-10000j}$

Example



$$v_I = V_i cos(\omega t), \qquad t > 0$$

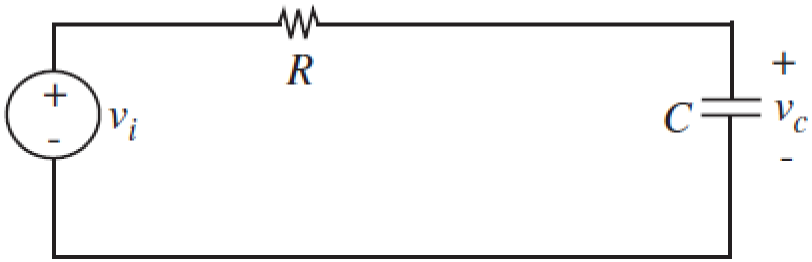
$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$
$$\omega = 100$$

Find $i(t)$ in steady state

- $\hat{I} = \frac{5\angle 0}{10000 - 10000j}$
- Polar divided by non polar, so convert bottom to polar
- $10000 - 10000j = 10000\sqrt{2}\angle\frac{-\pi}{4}$

Example

$$v_I = V_i cos(\omega t), \qquad t > 0$$



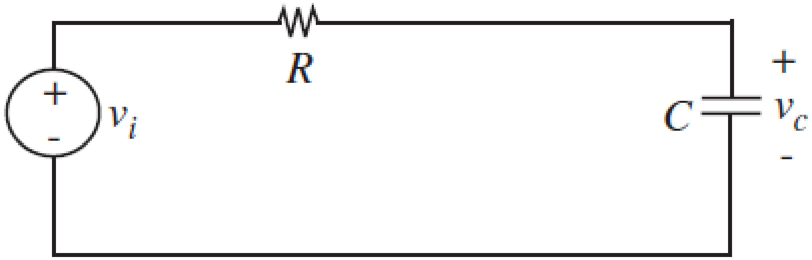
$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$
$$\omega = 100$$

Find $i(t)$ in steady state

- $\hat{I} = \frac{5\angle 0}{10000-10000j}$
- $10000 - 10000j = 10000\sqrt{2}\angle\frac{-\pi}{4}$
- So $\hat{I} = \frac{5\angle 0}{10000\sqrt{2}\angle\frac{-\pi}{4}} = \frac{1}{2000\sqrt{2}}\angle\frac{\pi}{4}$

Example

$$v_I = V_i cos(\omega t), \qquad t > 0$$



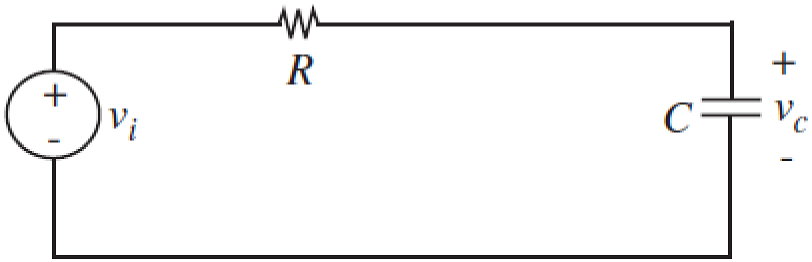
$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$
$$\omega = 100$$

Find $i(t)$ in steady state

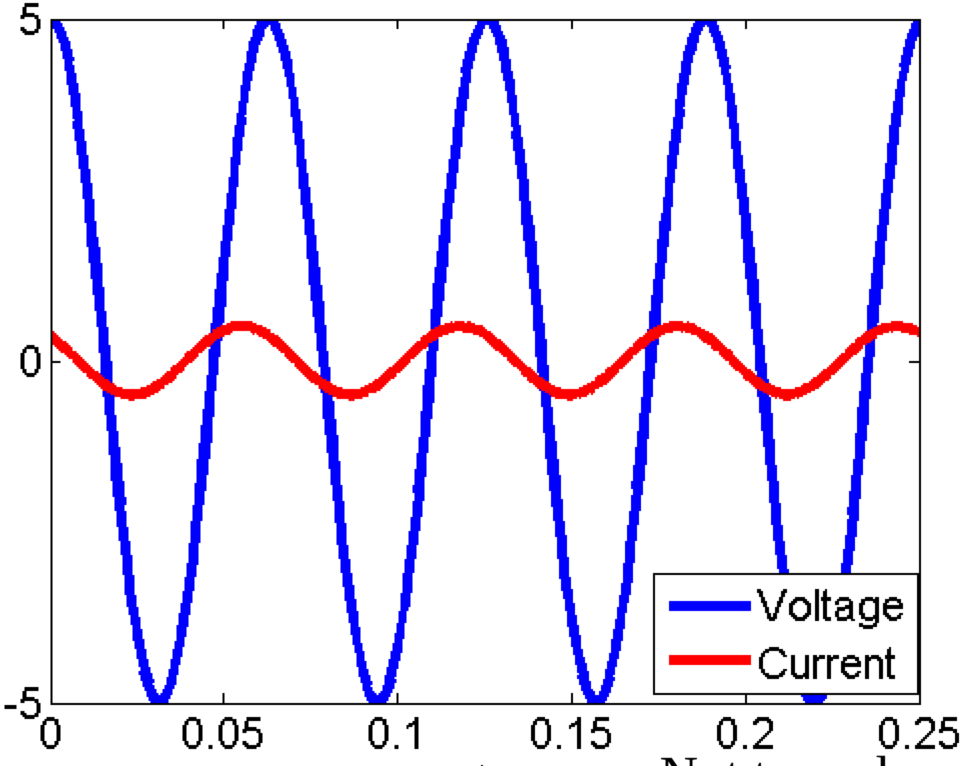
- $\hat{I} = \frac{1}{2000\sqrt{2}} \angle \frac{\pi}{4}$
- $i(t) = \frac{1}{2000\sqrt{2}} \cos(100t + \frac{\pi}{4})$ [in steady state]

Example

$$v_I = V_i cos(\omega t), \qquad t > 0$$



$$i(t) = \frac{1}{2000\sqrt{2}} \cos(100t + \frac{\pi}{4})$$



Not to scale

- Current has same shape as voltage
- Current is $10000\sqrt{2}$ times smaller than source voltage
- Current leads source voltage by $\frac{\pi}{4}$ radians or $\frac{\pi}{400}$ seconds

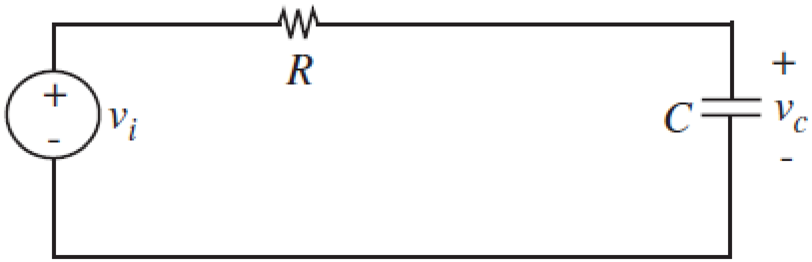
Harder Example

- On board

Filters

- Often, we'll want to build circuits which react differently based on different signal frequencies, e.g.
  - Splitting audio signals into low and high portions (for delivery to tweeter and subwoofer)
  - Removing noise of a particular frequency (e.g. 60 Hz noise or vuvuzela sound)
  - Removing signals except those at a certain frequency

Example Filter



$$v_I = V_i cos(\omega t), \qquad t > 0$$

$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$

Find $v_c(t)$ in steady state

- $\hat{V}_C = \frac{Z_C}{Z_C + Z_R} \hat{V}_I$
- $\hat{V}_C = \frac{-10^6 j/\omega}{-10^6 j/w + 10^5} \hat{V}_I$
- $\hat{V}_C = \frac{1}{1 + 0.1 jw} \hat{V}_I$

$$Z_C = -10^6 j/\omega$$
$$Z_R = 10^5$$

- **Transfer Function** $\boldsymbol{H(j\omega)}$

Transfer Functions

- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$
- Maps system input signal to system output signal
  - Plug an input voltage $A\cos(\omega t + \phi)$ into $\hat{V}_I$
  - Get an output voltage $A|H(j\omega)|\cos\left(\omega t + \phi + \angle H(j\omega)\right)$
    - Output is scaled and shifted in time
      - Scaling and shifting depend on frequency
    - Frequency is unchanged (linear system)
- Tells you how system will respond to any frequency, a.k.a. frequency response

Using a Transfer Function

- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$
- Suppose $v_i(t)$ is $3\cos(50t + \frac{\pi}{4})$
  - $\hat{V}_I = 3\angle\frac{\pi}{4}$
  - $H(j50) = \frac{1}{1+5j}$

$|H(j50)| = 1/\sqrt{26}$

$\angle H(j50) = -ArcTan[5/1]$

$= -1.37$

- Output phasor $\hat{V}_C$ is just $\hat{V}_I \times H(j50)$
  - $\hat{V}_C = \frac{3}{\sqrt{26}}\angle(\frac{\pi}{4} - 1.37)$
  - $v_c(t) = \frac{3}{\sqrt{26}}\cos(50t + \frac{\pi}{4} - 1.37)$

Using a Transfer Function (general)

- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$
- Suppose $v_i(t)$ is $3\cos(\omega t + \frac{\pi}{4})$
  - $\hat{V}_I = 3\angle\frac{\pi}{4}$
  - $H(jw) = \frac{1}{1+0.1j\omega}$

$|H(jw)| = 1/\sqrt{1 + 0.01\omega^2}$

$\angle H(j\omega) = -ArcTan[0.1\omega/1]$

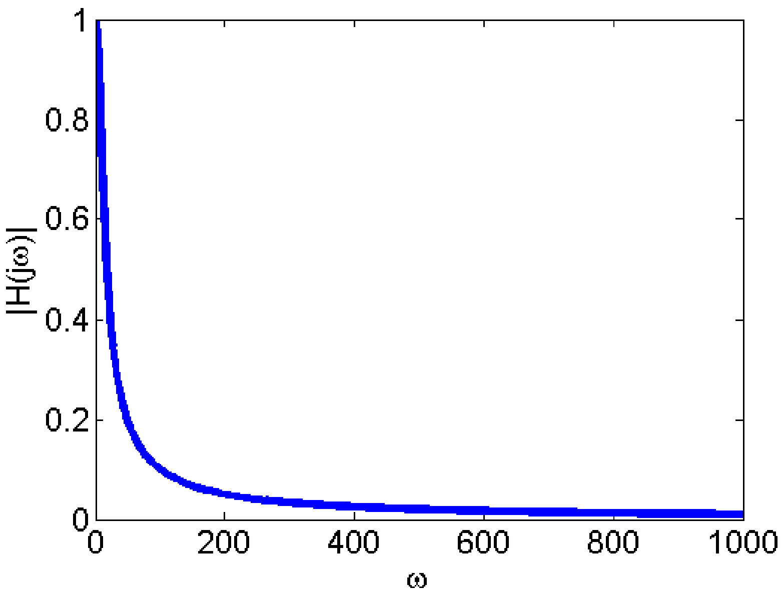
- Output phasor $\hat{V}_C$ is just $\hat{V}_I \times H(j50)$
  - $\hat{V}_C = \frac{3}{\sqrt{1+0.01\omega^2}}\angle(\frac{\pi}{4} - ArcTan\left[\frac{0.1\omega}{1}\right])$
  - $v_c(t) = \frac{3}{\sqrt{1+0.01\omega^2}}\cos(50t + \frac{\pi}{4} - ArcTan[\frac{0.1\omega}{1}])$

Bode Magnitude Plot

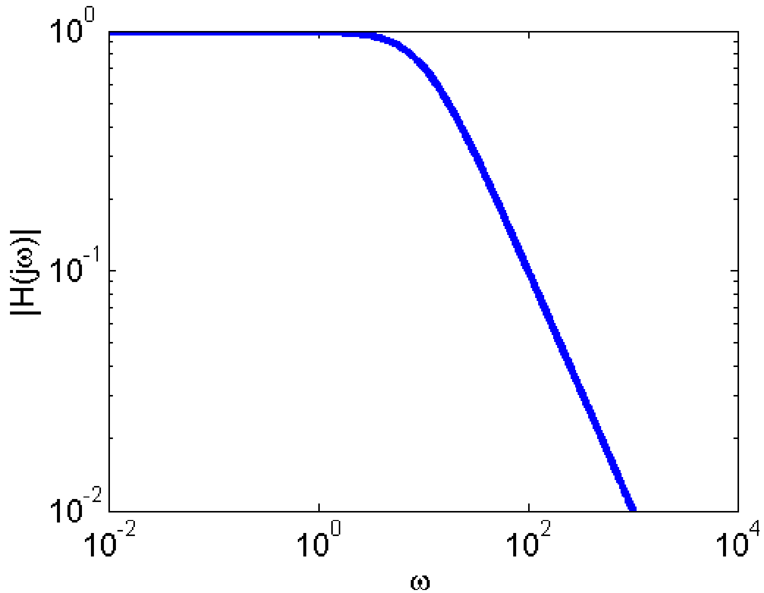
- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$

  $|H(j\omega)| = 1/\sqrt{1+0.01\omega^2}$

- Magnitude plot is just a plot of $|H(j\omega)|$ as a function of ω

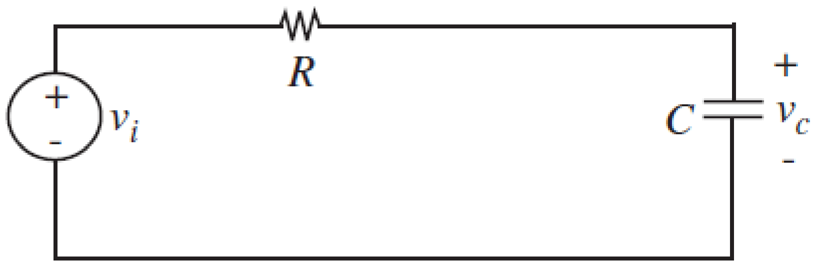


Linear Scale



Log Scale

Bode Magnitude Plot in Context of Circuit

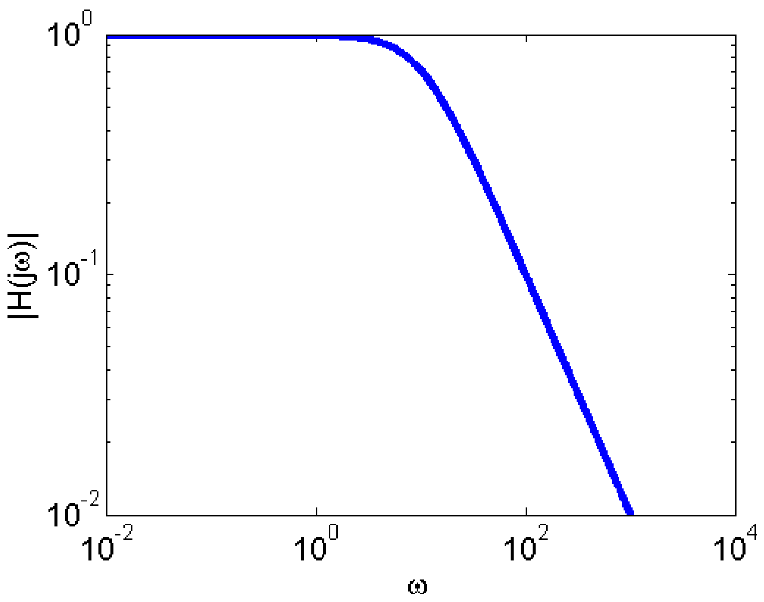


$$v_I = V_i cos(\omega t), \qquad t > 0$$

$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$

$$\hat{V}_C = \frac{1}{1 + 0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$$

$$|H(j\omega)| = 1/\sqrt{1 + 0.01\omega^2}$$



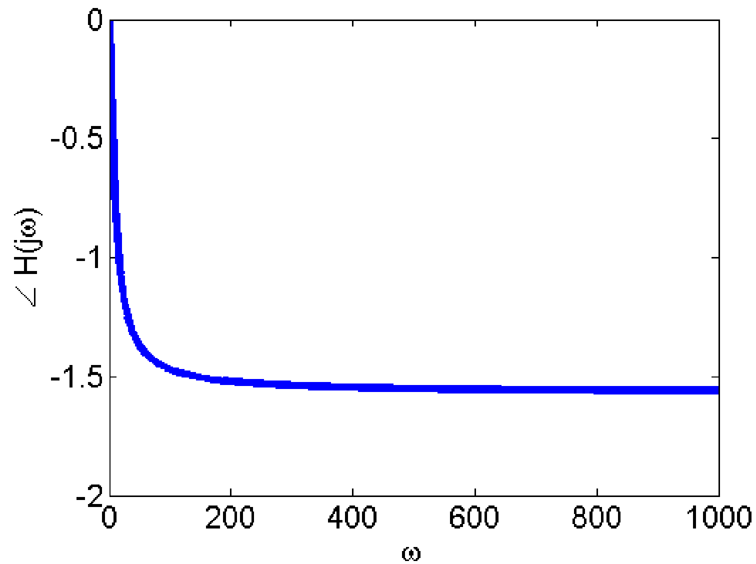
All frequencies below $w_c = 10$ get through pretty well. Above, that increasingly attenuated

Bode Phase Plot

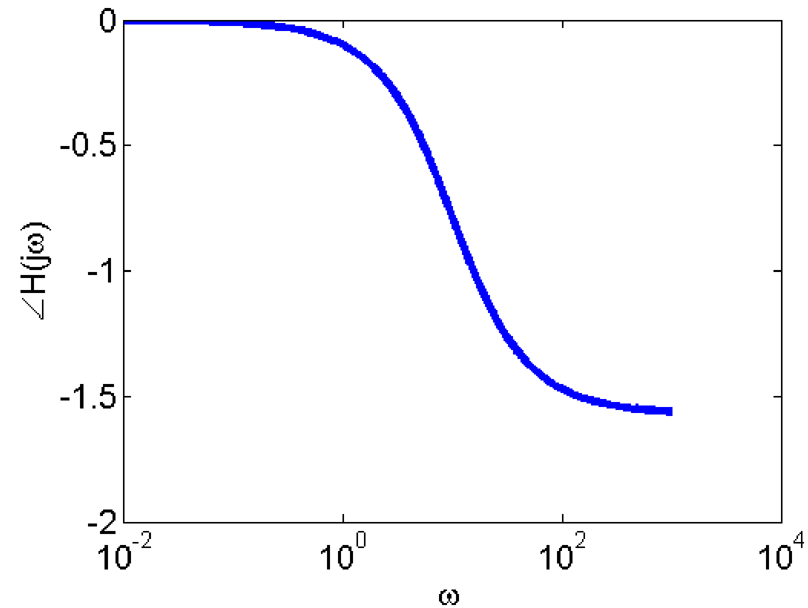
- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$

  $\angle H(j\omega) = -ArcTan[0.1\omega/1]$

- Phase plot is just a plot of $\angle H(j\omega)$ as a function of ω

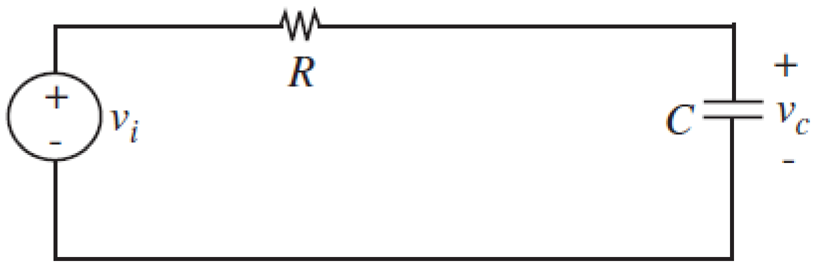


Linear Scale



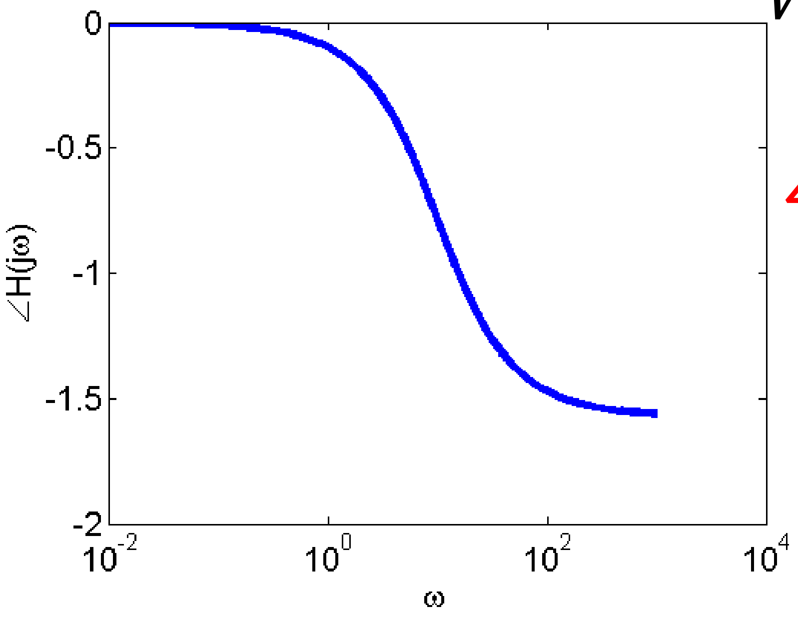
Semilog Scale

Bode Phase Plot in Context of Circuit



$$v_I = V_i cos(\omega t), \qquad t > 0$$

$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$



$$\hat{V}_C = \frac{1}{1 + 0.1jw}\hat{V}_I = H(j\omega)\hat{V}_I$$

$$\angle H(j\omega) = -ArcTan[0.1\omega/1]$$

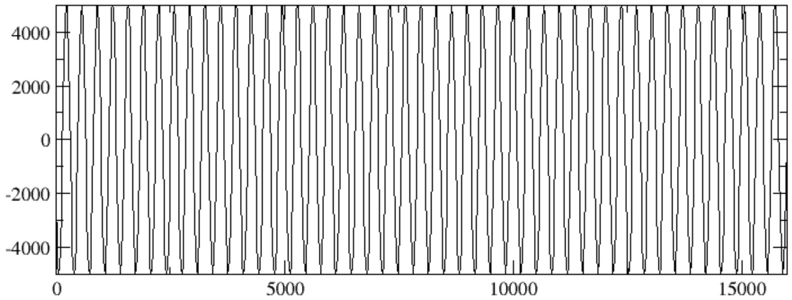
All frequencies below $w_c = 10$ move in time with the source, above that, v_c gets out of phase

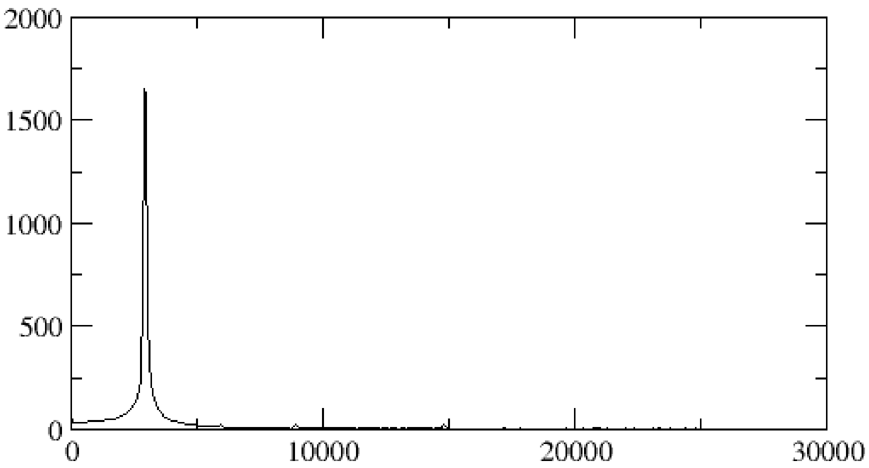
Frequency vs. Time Domain

- Almost always, our signals consist of multiple frequencies
- Examples:
  - Sound made when you press a buttons on a phone is two pure sine waves added together (DTMF)
  - Antennas on radio theoretically pick up ALL frequencies of ALL transmissions
- Using a technique known as the Fourier Transform, we can convert any signal into a sum of sinusoids
  - See EE20 for more details

Fourier Transform Example

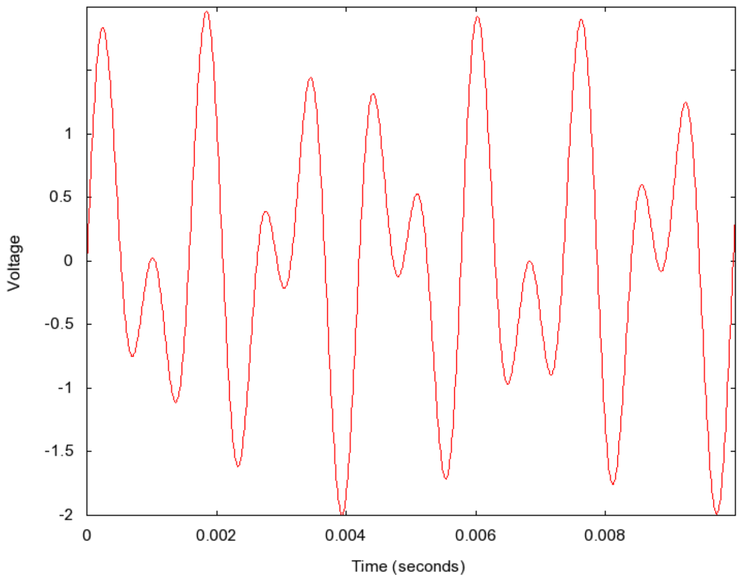
- If someone whistles a signal that is approximately $\sin(3000t)$, and we apply the Fourier Transform, then:

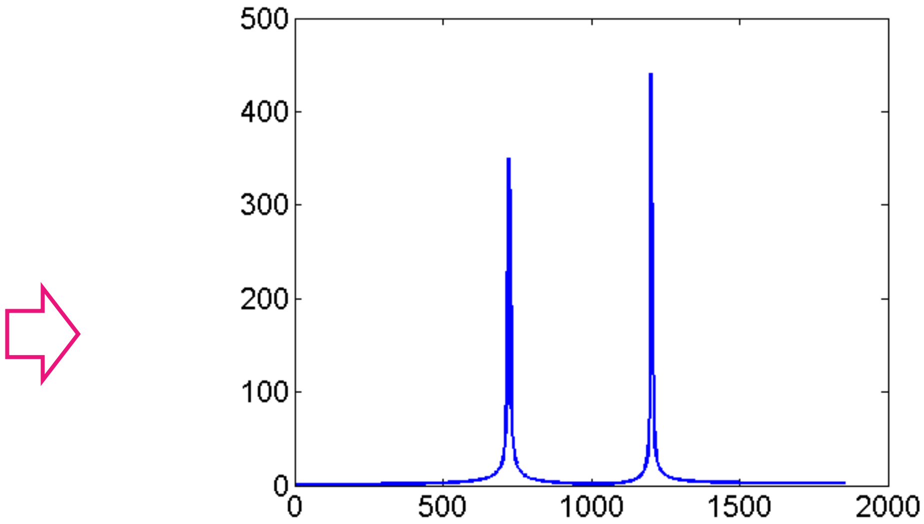




Fourier Transform Example

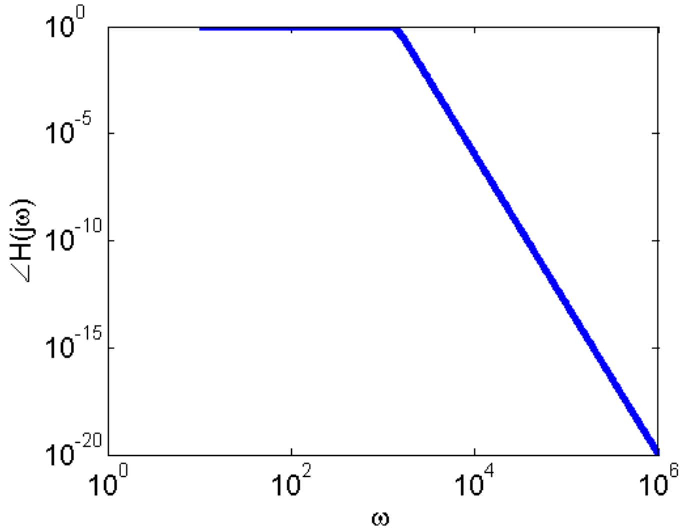
- The 1 button on a phone is just $v(t) = \sin\left(\frac{697}{2\pi} t\right) + \sin(\frac{1209}{2\pi} t)$

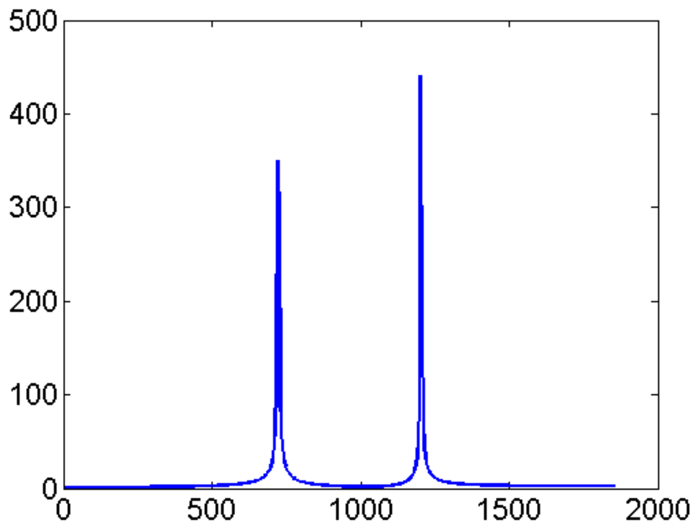




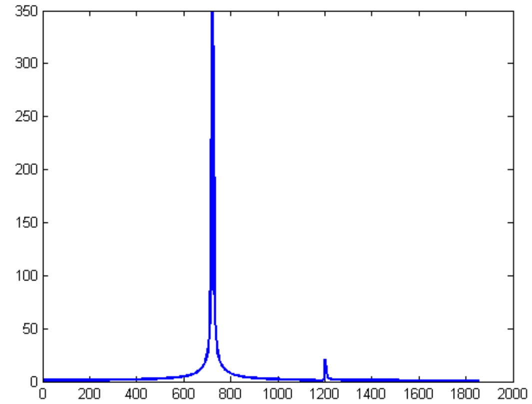
Fourier Transform Example

- If we apply a filter with the frequency response on the left to the signal on the right





Then we'll get:



Types of Filters

- Passive Filters
  - Filters with no sources (i.e. just R, L, and C)
  - Don’t require power source
  - Scale to larger signals (no op-amp saturation)
  - Cheap
- Active Filters
  - Filters with active elements, e.g. op-amps
  - More complex transfer function
    - No need for inductors (can be large and expensive, hard to make in integrated circuits)
    - More easily tunable
  - Response more independent of load (buffering)

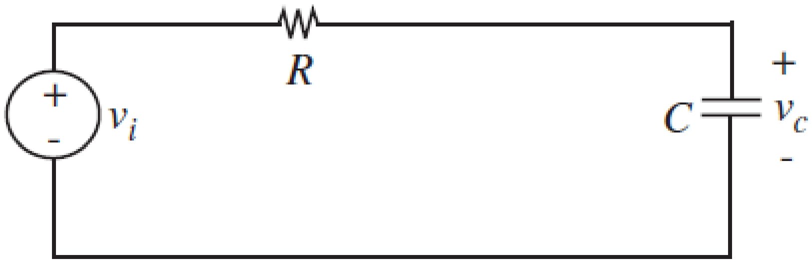
Filter Examples

- On board

Manually Plotting

- In this day and age, it is rarely necessary to make our Bode plots manually
- However, learning how to do this will build your intuition for what a transfer function means
- Manual plotting of bode plots is essentially a set of tricks for manually plotting curves on a loglog axis
- We will only teach a subset of the full method (see EE20 for a more thorough treatment)

Example Filter



$$v_I = V_i cos(\omega t), \qquad t > 0$$

$$R = 10{,}000\Omega$$
$$C = 1\mu F$$
$$V_i = 5V$$

Find $v_c(t)$ in steady state

- $\hat{V}_C = \frac{1}{1+0.1jw}\hat{V}_I$
- $\hat{V}_C = \frac{1}{\sqrt{1+0.01\omega^2}\angle ArcTan[0.1\omega]}\hat{V}_I$
- $\hat{V}_C = \frac{1}{\sqrt{1+0.01\omega^2}}\angle - ArcTan[0.1\omega]\ \hat{V}_I$
- Intuitive plot on board
- More thorough algorithm next time

Extra Slides

Why do equivalent Impedances work?

- Components with memory just integrate or take the derivative of $e^{q_1 t}$, giving scaled versions of the same function
  - This is unlike forcing functions like t^3 or $\cos(\omega t)$
  - This allows us to divide by the source, eliminating t from the problem completely
  - Left with an algebra problem
  - [For those of you who have done integral transforms, this whole process can be thought of as just using Laplace/Fourier transforms]