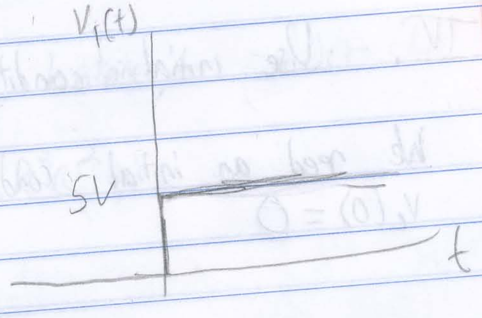
A more typical source is a step input.

Think of as a source being switched on or a battery suddenly connected



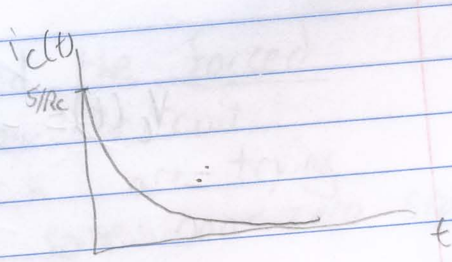
Try the exercise we just did, but now with the source above assuming $V_c(0)=0$

You should get: $V_c(t) = 5 - 5e^{-t/RC}$

At $t=0$ $V_c(0)=0$

$t=\infty$ $V_c(\infty)=0$

$$i_c(t) = C \cdot V_c'(t) = \frac{5}{RC} e^{-t/RC}$$



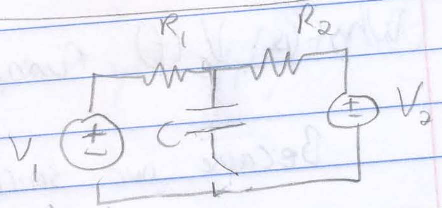
$i_c(0) = 5/RC$

$i_c(\infty) = 0$

Acts like a short at $t=0$, open at $t=\infty$

Lec 9 pg 1

Warm up: How would you solve?



Assume $V_1 = V_2 = \begin{cases} 0, & t<0 \\ V_k, & t\geq 0 \end{cases}$

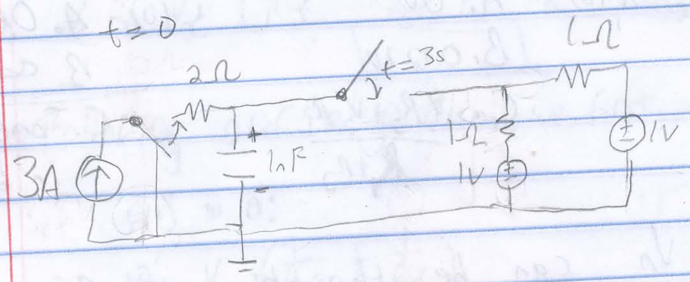
Node voltage: $\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + C\frac{dV_c(t)}{dt} = 0$

$V_c(0) = 0.127$

Standard Form: $-V_c' = V_c\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) + \frac{V_1}{CR_1} + \frac{V_2}{CR_2}$

Michael
Annie
Megan
Will

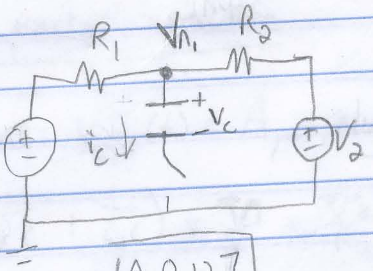
$i = C\frac{dv}{dt}$



Find $V_c(t)$, if $V_c(t)=0$ if $t<0$.

Before we go through and solve this ODE let's think about what happens to V_c, i_c at $t=-1$, $t=0$, and $t=\infty$

$t=-1$:



At $t=-1$, and indeed all times before time 0, no current can flow through C. Thus V_c must be constant!

iQ: What is $V_c(-1)$? $\boxed{A.\ 0.127}$ B. 0V C. Some other value

$V_c(-1) = 0.127$

But wait! Won't V_{n_1} be some value other than 0.127?

Yes, but the bottom side of the capacitor is not ground, so V_c is not same as V_{n_1}!

Since V_{n_1} is an interesting quantity, let's consider it, too, in addition to V_c, i_c?

iQ: What is $V_{n_1}(-1)$? $\boxed{A.\ \frac{V_1 \cdot R_2 + V_2 \cdot R_1}{R_1+R_2}}$ B. $\frac{V_1 \cdot R_1 + V_2 \cdot R_2}{R_1+R_2}$

iQ: Now what are $V_c(0)$, $i_c(0)$, and $V_{n_1}(0)$?

$V_c(0)$: A. 0V
$\boxed{\text{B. } 0.127V}$
C. $\frac{V_1R_2+V_2R_1}{R_1+R_2}$

$V_{n_1}(0)$: A. 0V
$\boxed{\text{B. } 0.127V}$
C. $\frac{V_1R_2+V_2R_1}{R_1+R_2}$

$i_c(0)$: A. 0A
B. ∞A
$\boxed{\text{C. Something in between}}$

The sudden drop in V_{n_1} can be thought of as a sudden pressure drop due to a hole in a pipe when the capacitor appears and starts sucking up charge.

What are $V_c(\infty)$, $i_c(\infty)$, and $V_{n_1}(0)$?

$V_c(\infty)$ A. 0V
B. 0.127V
$\boxed{\text{C. } \frac{V_1R_2+V_2R_1}{R_1+R_2}}$
D. Need to solve ODE

$V_{n_1}(\infty)$ SAME

$i_c(0)$: A. 0A
B. ∞ A
C. Need to solve ODEs.

$V_I = 0.127A$

$$V_F = \frac{V_1R_2+V_2R_1}{R_1+R_2}$$

What happens in between will require ODEs.

0. $V_C' = V_C\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) + \overbrace{\frac{V_1}{R_1} + \frac{V_2}{R_2}}^{f(t)}$ – the SFODE

I. Following our algorithm, we first find homogeneous solution

Set $f(t)$ to 0:

$$V_C' = -V_C\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)$$

$$S = -\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)$$

Homogeneous solution: $V_{C,H}(t) = K_1 e^{st} = K_1 e^{-\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)t}$

(also known as natural response)

Note that the natural response is just the equation for voltage decay

II. Now find particular solution

Assume $V_{C,P}(t) = A$, plug into SFODE.

$$0 = -A\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) + \frac{V_1}{R_1} + \frac{V_2}{R_2}$$

$$A = \frac{\frac{V_1}{CR_1} + \frac{V_2}{CR_2}}{\frac{1}{R_1C} + \frac{1}{R_2C}} = \frac{\frac{V_1}{CR_1} + \frac{V_2}{CR_2}}{\frac{R_2 + R_1}{R_1R_2C}} = \frac{\frac{R_2V_1 + R_1V_2}{CR_1R_2}}{\frac{R_1 + R_2}{R_1R_2C}}$$

Note that the forced response is just the steady state voltage

Particular solution $A = \frac{(R_2V_1 + R_1V_2)}{R_1 + R_2}$

III. $V_C(t) = K_1 e^{-\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)t} + \frac{(R_2V_1 + R_1V_2)}{R_1 + R_2}$ (Complete solution)

IV. $V_C(0) = 0.127 \Rightarrow K_1 = -\frac{(R_2V_1 + R_1V_2)}{R_1 + R_2} + 0.127$

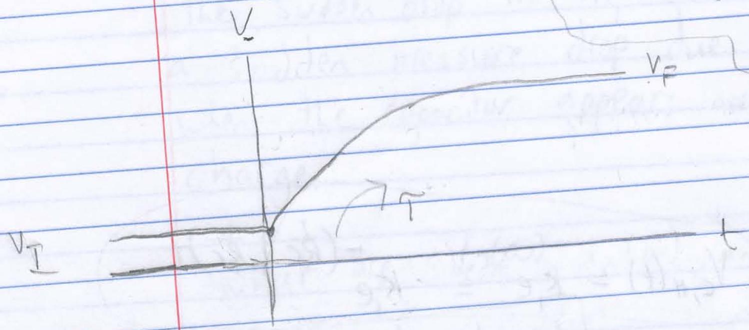
(Find unknown constant K_1)

Creating variables

$$V_F = \frac{(R_2 V_1 + R_1 V_2)}{R_1 + R_2} \quad \text{similar} \quad \tau = 1\Big/\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)$$

$V_I = 0.127$

$$\boxed{V_c(t) = \underbrace{(V_I - V_F)e^{-t/\tau}}_{\text{natural response}} + \underbrace{V_F}_{\text{Forced response}}}$$



We refer to τ as the "time constant." It tells us how slow the system changes. If τ is large, then the capacitor will take a very long time to charge (and discharge).

Physically, it represents the time it takes for the exponential to reach 63.2% $(1 - 1/e)$ of its final value. So if the capacitor were charging from 0V to 1000V, it would take τ seconds to reach 632 V (assuming we're measuring time in seconds).

It's identical to the concept of half-life, except it's 63.2% instead of 50%.

τ depends on the circuit that the element with memory sees. For a RC circuit, τ will just be $\tau = \frac{1}{R_{eq} C}$ where

R_{eq} is the resistance of the circuit looking out from the capacitor terminals (not source terminals).

For an RL circuit, it's just $\tau =$ L/R

This brings us to the ultra-fast way of doing 1st order circuits, which the book calls "the intuitive method."

You may have noticed a pattern in the 3 problems we've done so far. When solving for a capacitor voltage in an RC circuit, if we have only DC sources (constant sources), then:

pretty easy to prove.

$$V_c(t) = (V_I - V_F)\, e^{-t/\tau} + V_F$$

When $t=0$ $\quad V_c(0) = (V_I - V_F) + V_F = V_I$

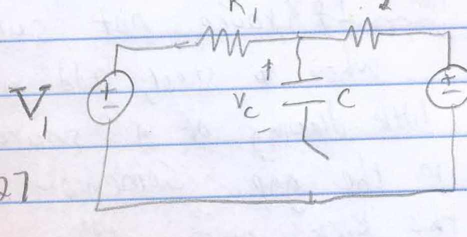
" $\quad t=\infty \quad V_c(\infty) = (V_I - V_F)\cdot 0 + V_F = V_F$

Where $V_I = V_c(0)$, $\quad V_F = V_c(\infty)$, $\quad \tau = \frac{1}{R_{eq} C}$

For an RL circuit, it's just:

$$i_L(t) = (i_I - i_F)\, e^{-t/\tau} + i_F \qquad \begin{aligned} i_I &= i_L(0) \\ i_F &= i_L(\infty) \\ \tau &= L/R_{eq} \end{aligned}$$

Examples:

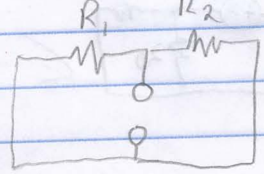


$V_c(0) = 0.127$

We know from iClicker questions that $V_I = 0.127$

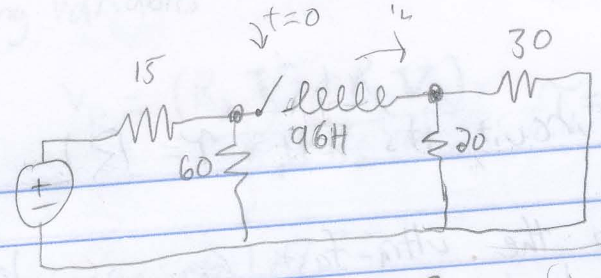
$$V_F = \frac{V_1 \cdot R_2 + V_2 R_1}{R_1 + R_2}$$

To find R at Cap terminals, we:



$$\tau = \frac{1}{1/R_1 + 1/R_2}$$

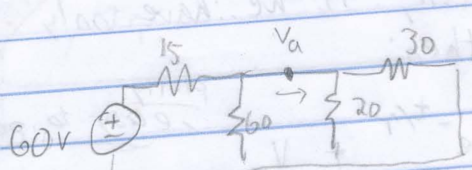
Done in 30 seconds by inspection!



What is $i_L(0)$? $\boxed{i_L(0) = 0A}$

What is $i_L(\infty)$?

Well after a looong time, v_L will be 0 (acts like a short), so:



$$V_a = \frac{(30 \| 20 \| 60)}{(30\|20\|60)+15} \cdot 60V$$

$$\frac{30 \cdot 20}{30+20} = \frac{600}{50} = 12 \Rightarrow \frac{12 \cdot 60}{12+60} = \frac{720}{72} = 10\,\Omega$$

$$= \frac{10\Omega}{10\Omega + 15\Omega} \cdot 60V$$

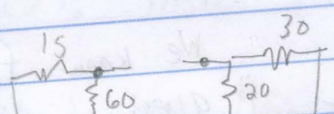
$$= \tfrac{2}{5} \cdot 60V = 24V$$

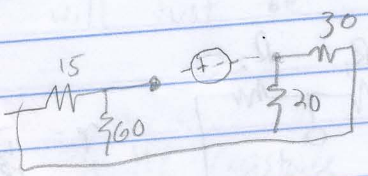
10/25

$$\boxed{i_F = \frac{24V}{30\|20\Omega} = \frac{24V}{12\Omega} = 2A}$$

2/5

What is τ? It's $\tau = L/R_{eq}$ where R_{eq} is equivalent resistance seen by inductor.





If you're not sure where to start, add a little drawing of a source in the gap... Somehow this kicks our pattern recognition algorithm in our brain into gear.

Clearly R_{eq} is just $(15 \| 60) + (20 \| 30) = \frac{900}{75} + \frac{600}{50} = 12 + 12 = 24\Omega$

Thus τ is just

$$\tau = \frac{96H}{24\Omega} = 4s$$

Giving us: $i_L(t) = (i_I - i_F)e^{-t/4} + i_F = -2e^{-t/4} + 2$

No homogeneous solutions, no particular solutions, just some relatively fast circuit analysis and you're done.

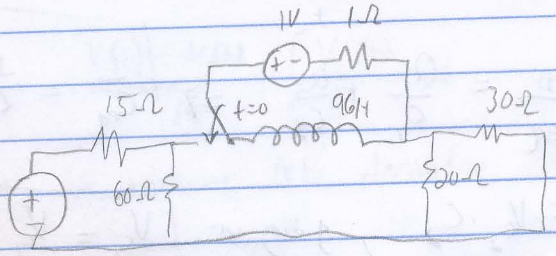
Q: ~~How~~ long does it take to reach 86.5% of 2A?

$$0.865 \cdot 2 = -2e^{-t/4} + 2$$
$$0.865 = -e^{-t/4} + 1$$
$$-0.135 = e^{-t/4}$$
$$\ln(-0.135) = -t/4$$
$$-4\ln(-0.135) = t$$
$$-4 \cdot (-2) = t$$

$t = 8s$ (2 time constants).

Q: How would you change the circuit so that the only difference was $i_L(0) = 1A$?

Hint: you can have two switches if you want.



$i_L(t) = -e^{-t/4} + 2.$

$$\int_0^t x^2\,dx = \frac{x^3}{3}\Big| \Rightarrow \frac{t^3}{3}$$

Capacitors in series: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \ldots + \frac{1}{C_n}$

$$\int_0^t \sin(x)\,dx = -\cos(x)\Big|$$

$$-\cos(t) + 1$$

" in parallel: $C_{eq} = C_1 + \ldots + C_n$

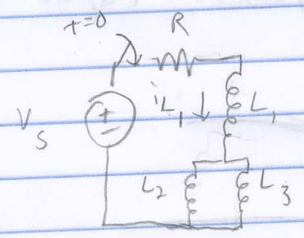
$\sin(t) = \sin(t)$

Inductors in series: $L_{eq} = L_1 + \ldots + L_n$

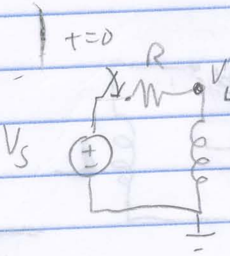
" in parallel $1/L_{eq} = 1/L_1 + \ldots + 1/L_n$

Arjun

Example: Write SFODE for inductor current below:



$\Rightarrow$



$L_1 + (L_2 \| L_3) = L_{eq}$

$$V_L = L\frac{di_L}{dt}$$

easy way KVL $\quad -V_s(t) + i_L(t)\cdot R + L\,i_L'(t) = 0$

$$i_L'(t) = -\frac{i_L(t)\cdot R}{L} + \frac{V_s(t)}{L}$$

Or node voltage:

$$\frac{V_L(t) - V_s(t)}{R} + \int_0^t \frac{V_L(t)}{L}\,dt$$

$V_s' = 0$

Integral is annoying, let's derivative it away:

$$\frac{V_L'(t) - V_s'(t)}{R} + \frac{V_L(t)}{L} = 0$$

$$i_L = \frac{V_s - V_L}{R}$$

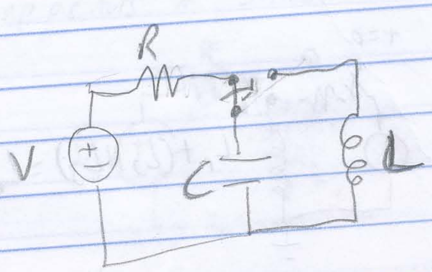
$$V_L'(t) = -\frac{V_L(t)\cdot R}{L} + V_s'(t)$$

$v_L = v_S - Ri_L$ $\qquad v_L' = v_S' - Ri_L'$

$$\cancel{v_S'(t)} - Ri_L' = \frac{-R\cdot(v_S - Ri_L(t))}{L} + \cancel{v_S'(t)}$$

$$i_L' = -\frac{R}{L} i_L(t) + \frac{v_S(t)}{L}$$

LC circuits



In left position capacitor charges.

In the right position, we have to do some work.

Node voltage gives us: $\qquad i_C = CV_C' \qquad V_L = L i_L' \qquad i_L = \int \frac{V_L}{L}$

$$CV_C' + \int \frac{V_L}{L} = 0$$

$$CV_C'' + \frac{V_L}{L} = 0 \qquad \text{Note } V_L = V_C \text{ (parallel)}$$

$$CV_C'' + \frac{V_C}{L} = 0.$$

How do we solve? Well, this is a 2^{nd} order ODE so we use a new procedure (see handout).