EECS 127/227AT **Fall 2020**
Discussion 1

1 A quick proof of the Cauchy-Schwarz inequality

(a) Given vectors $u, v \in \mathbb{R}^n$, show that

$$\|u\|_2^2 + \|v\|_2^2 \geq 2u^\top v.$$

Hint: Start by expanding $\|u - v\|_2^2$.

(b) Assume u and v are nonzero. Apply the result from part (a) of this problem to the normalized vectors $u' := u/\|u\|_2$ and $v' := v/\|v\|_2$ to arrive at the conclusion that

$$u^\top v \leq \|u\|_2 \|v\|_2.$$

Note that this also holds if one or both of the vectors u and v is the zero vector, but it is obvious in these cases.

(c) The standard statement of the Cauchy-Schwarz inequality, which follows immediately from the result in part (b), is

$$|u^\top v| \leq \|u\|_2 \|v\|_2 \quad \forall u, v \in \mathbb{R}^n.$$

In fact one can even strengthen this further to the sometimes strictly stronger statement that $\sum_{i=1}^n |u_i||v_i| \leq \|u\|_2 \|v\|_2$ by simply replacing u by the vector with coordinates $|u_i|$, which has the same ℓ_2 norm as u, and v by the vector with coordinates $|v_i|$, which has the same ℓ_2 norm as v.

Now, consider a fixed, nonzero $u_0 \in \mathbb{R}^n$. For what v is the inequality an equality? Your answer should contain u_0.

2 Monotonicity of ℓ_p norms.

(a) First, we'll show that for $a_1, \ldots, a_n \in \mathbb{R}_{>0}$ (i.e., each a_i is a positive real number) and $0 < m < 1$,

$$\left(\sum_{i=1}^{n} a_i\right)^m \leq \sum_{i=1}^{n} a_i^m,$$

where the inequality is strict if $n \geq 2$.

i. Define

$$f(a_1, \ldots, a_n) \coloneqq \left(\sum_{i=1}^{n} a_i\right)^m - \sum_{i=1}^{n} a_i^m.$$

We'll show $f(a_1, \ldots, a_n) \leq 0$ to prove the result. To start, compute $\partial f(a_1, \ldots, a_n)/\partial a_j$, where $j \in \{1, ..., n\}$.

ii. Using the result of the previous part, show that $\partial f(a_1, \ldots, a_n)/\partial a_j \leq 0$ for all j, with strict inequality when $n \geq 2$. Remember that we assume $a_1, ..., a_n \in \mathbb{R}_{>0}$.

iii. Use the result of the previous part to conclude that $f(a_1, \ldots, a_n) \leq 0$ when $a_1, ..., a_n \in \mathbb{R}_{>0}$, with strict inequality when $n \geq 2$, thus proving the original claim.

Hint: To start, it may help to compute $f(0, \ldots, 0)$.

(b) For $x \in \mathbb{R}^n$, recall the definition of the ℓ_p norm:

$$\|x\|_p \coloneqq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Using the result of part (a), conclude that for $1 \leq q < p < \infty$ and nonzero $x \in \mathbb{R}^n$ having at least two nonzero coordinates, we have

$$\|x\|_p < \|x\|_q.$$

3 Gram-Schmidt process

Let $\mathcal{P}$ be the subspace in $\mathbb{R}^3$ spanned by the vectors $x_1 := \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^\top$ and $x_2 := \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^\top$. In the first two parts of the problem we will establish that $\mathcal{P}$ is a plane and then we will proceed with the rest of the problem.

(a) Find the angle between the vectors x_1 and x_2.

(b) Show that x_1 and x_2 are linearly independent.

Hint: Think about the situation geometrically – what does the angle between the vectors tell us about their linear (in)dependence?

(c) Find an orthonormal basis $B_{\mathcal{P}}$ for the plane $\mathcal{P}$ using the Gram-Schmidt process.

(d) Extend $B_{\mathcal{P}}$ to B, an orthonormal basis for $\mathbb{R}^3$.

(e) Use B to find the distance of the vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ from the plane $\mathcal{P}$.

4 Extrema of the inner product over an Euclidean ball

Let $y \in \mathbb{R}^n$ be a given nonzero vector, and let $\mathcal{X} := \{x \in \mathbb{R}^n : \|x\|_2 \leq r\}$, where r is some given strictly positive number.

(a) Determine the optimal value p_1^* and the optimal set (i.e., the set of *all* optimal solutions) of the problem $\min\limits_{x \in \mathcal{X}} |y^\top x|$.

(b) Determine the optimal value p_2^* and the optimal set of the problem $\max\limits_{x \in \mathcal{X}} |y^\top x|$.

(c) Determine the optimal value p_3^* and the optimal set of the problem $\min\limits_{x \in \mathcal{X}} y^\top x$.

(d) Determine the optimal value p_4^* and the optimal set of the problem $\max\limits_{x \in \mathcal{X}} y^\top x$.