EECS 127/227AT **Fall 2020**
Discussion 2

1 Symmetric Matrices and Quadratic Forms

(a) A general quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ can be written as

$$f(x) = x^T Ax + x^T b + c,$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Note that $x^T Ax$ is a number, so if we take the transpose we get

$$x^T Ax = x^T A^T x,$$

from which we get

$$x^T Ax = x^T (\frac{A + A^T}{2})x,$$

so, when discussing a quadratic function, there is no loss of generality in assuming that A is symmetric. In the special case where $b = 0$ and $c = 0$, and assuming A is symmetric, we get the quadratic function $x^T Ax$ on $\mathbb{R}^n$. Such a function is called a *quadratic form*.

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, i.e. $A = A^T$. In class we have shown that the eigenvalues of A are real and that after an orthogonal change of basis A can be expressed as a diagonal matrix, i.e. we can write $A = U^T \Lambda U$ where U is an orthogonal matrix and Λ is a diagonal matrix.

In this problem we will understand the quadratic form associated to a symmetric positive semidefinite matrix A by considering the set

$$\mathcal{E} := \{x \in \mathbb{R}^n : x^T Ax \leq 1\}.$$

Note that if A were the identity matrix then $\mathcal{E}$ would be the unit ball in $\mathbb{R}^n$, so in general $\mathcal{E}$, as defined above, can be considered the analog of the unit ball for the quadratic form defined by $A \in \mathbb{S}^n_+$.

Sketch the set $\mathcal{E}$ for the following matrices and think of how the eigenvalues and eigenvectors of A relate to the shape of the set $\mathcal{E}$.

i. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Note that here it is obvious that A is symmetric and positive semidefinite.

ii. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
Note that here it is still obvious that A is symmetric and positive semidefinite.

iii. $A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$
Here A is symmetric but it is not immediately obvious that it positive semidefinite. For this we need to compute the eigenvalues. Here this is easy to do by computing the trace (which is the sum of the eigenvalues) and the determinant (which is the product of the eigenvalues). From this we can conclude that A is positive semidefinite, since the trace is 3 and the determinant is 2, so the eigenvalues must be 2 and 1.

(b) Let $A \in \mathbb{S}_+^n$ (i.e. A is a symmetric positive semidefinite matrix), $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Consider the following minimization problem:

$$p^* = \min_{x \in \mathbb{R}^n} x^T A x + b^T x + c.$$

We will now study this problem in some examples.

Find the optimal value of this optimization problem, i.e. p^*, for the following cases:

(i) $A = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$, $b = 0$, $c = 0$.
You should check that here we have $A \in \mathbb{S}_+^n$.

(ii) $A = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$, $b = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, $c = 0$.

2 Cauchy matrix

(a) Let $v \in \mathbb{R}^n$. Let $A := vv^T$. Note that $A \in \mathbb{R}^{n \times n}$.

Recall that $\mathbb{S}^n$ is our notation for the set of symmetric matrices in $\mathbb{R}^{n \times n}$, and that $\mathbb{S}^n_+$ is our notation for the subset of $\mathbb{S}^n$ comprising symmetric positive semidefinite matrices.

Show that $A \in \mathbb{S}^n_+$. What is the rank of A?

(b) Let $A, B \in \mathbb{S}^n_+$. Show that $A + B \in \mathbb{S}^n_+$.

(c) Let $(A_k \in \mathbb{S}^n_+)_{k \geq 1}$ be a sequence of symmetric positive semidefinite $n \times n$ matrices. Suppose that $\lim_{k \to \infty} A_k = A$ in the sense that the individual entries $(A_k)_{ij}$ converge to $(A)_{ij}$ as $k \to \infty$ for $1 \leq i, j \leq n$.

Show that $A \in \mathbb{S}^n_+$.

(d) Let $\alpha_1, \ldots, \alpha_n$ be n strictly positive real numbers. The associated *Cauchy matrix* is the matrix $A \in \mathbb{R}^{n \times n}$ whose (i, j) entry is $\frac{1}{\alpha_i + \alpha_j}$, i.e.

$$A := \begin{bmatrix} & \vdots & \\ \cdots & \frac{1}{\alpha_i + \alpha_j} & \cdots \\ & \vdots & \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Show that $A \in \mathbb{S}^n_+$.

Hint:

$$\frac{1}{\alpha_i + \alpha_j} = \int_0^\infty e^{-(\alpha_i + \alpha_j)t} dt.$$

3 Adding a dyad

(a) Let $A \in \mathbb{S}^n$ (note that positive semidefiniteness is not assumed in this part of the problem).

Show that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are orthogonal complements of each other.

(b) Let $A \in \mathbb{S}^n_+$ and $x \in \mathbb{R}^n$.

Show that

$$x \in \mathcal{N}(A) \Leftrightarrow x^T A x = 0.$$

(c) Let $v, w \in \mathbb{R}^n$. Show that

$$\mathcal{R}(vv^T) \subseteq \mathcal{R}(vv^T + ww^T).$$

Hint: When v is nonzero, write $w = (w - \frac{w^T v}{\|v\|_2^2} v) + \frac{w^T v}{\|v\|_2^2} v$.

(d) Let $A \in \mathbb{S}^n_+$ and $v \in \mathbb{R}^n$. From part (b) of preceding problem, we know that $A + vv^T \in \mathbb{S}^n_+$.

What possible values can $\text{rank}(A + vv^T) - \text{rank}(A)$ take? Explain under what scenarios each of these values results.