EECS 127/227AT **Fall 2020**
Discussion 4

1 Convexity of Sets

A set $C \subseteq \mathbb{R}^n$ is called *convex* if the line segment between any two points in C lies in C, i.e., for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have,

$$\theta x_1 + (1-\theta)x_2 \in C.$$

(a) Show that the following sets are convex:

i A vector subspace $\mathcal{V}$ of $\mathbb{R}^n$.

ii A hyperplane $\mathcal{L}$ in $\mathbb{R}^n$, given by $\mathcal{L} := \{x \ : \ a^\top x = b\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

iii A halfspace $\mathcal{H}$ in $\mathbb{R}^n$, given by $\mathcal{H} := \{x \ : \ a^\top x \leq b\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

iv A norm-ball $\mathcal{B}$ in $\mathbb{R}^n$, given by $\mathcal{B} := \{x \ : \ \|x - x_c\| \leq r\}$, where $x_c \in \mathbb{R}^n$, $r \geq 0$ and $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

(b) Operations that preserve convexity of sets:

i **Intersection**

1. Show that convexity is preserved under intersection, i.e. if S_1 and S_2 are convex subsets of $\mathbb{R}^n$, then $S := S_1 \cap S_2$ is convex.

2. Show that a polyhedron is convex. A *polyhedron* is the solution set of a finite number of linear inequalities. For example $P := \{x \in \mathbb{R}^2 \mid x_2 \leq x_1, x_1 \geq 0, x_1 \leq 1\}$ is a polyhedron.

ii **Mapping under an affine function**
A function $f \ : \ \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if it is a sum of a linear function and a constant, i.e.

$$f(x) = Ax + b,$$

for $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

1. Prove that if $S \subseteq \mathbb{R}^n$ is convex, then the image of S under f,

$$f(S) := \{f(x) \ : \ x \in S\},$$

which is a subset of $\mathbb{R}^m$, is convex.

Example: Let $e_i, 1 \leq i \leq n$ denote the standard basis vectors in $\mathbb{R}^n$. As an example of what we have just shown, the projection of a convex set in $\mathbb{R}^n$ onto the span of any subset of the standard basis vectors is convex. E.g. if $S \subseteq \mathbb{R}^3$ is convex, then $T := \{(x_1, x_2) \in \mathbb{R}^2 \ : \ (x_1, x_2, x_3) \in S \text{ for some } x_3 \in \mathbb{R}\}$ is convex.

2. Show that the inverse image of a convex set $C \subseteq \mathbb{R}^m$ under the affine function $f(x) = Ax + b$ is convex. Here, the inverse image of C is defined to be the subset of $\mathbb{R}^n$ given by

$$f^{-1}(C) := \{x \;:\; f(x) \in C\}.$$

2 Convexity of Functions

Recall the following definitions and facts involving convex functions:

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if $\mathrm{dom}(f)$ is a convex set and if, for all $x, y \in \mathrm{dom}(f)$ and $0 \le \theta \le 1$, we have

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y). \tag{1}$$

  The function f is called *strictly convex* if the inequality in (1) is strict whenever $\theta \in (0,1)$ and $x \neq y$.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *concave* if $\mathrm{dom}(f)$ is a convex set and if, for all $x, y \in \mathrm{dom}(f)$ and $0 \le \theta \le 1$, we have

$$f(\theta x + (1-\theta)y) \ge \theta f(x) + (1-\theta) f(y).$$

  The function f is called *strictly concave* if the inequality is strict when $\theta \in (0,1)$ and $x \neq y$.

- A function f is concave if and only if $-f$ is convex and strictly concave iff $-f$ is strictly convex. An affine function is both convex and concave.

It is useful to keep in mind several alternative characterizations of convexity:

- *First order condition*
  Suppose f is differentiable at every point in its domain (in particular, for this to make sense, $\mathrm{dom}(f)$ would need to be an open set). Then f is convex if and only if we have

$$f(y) \ge f(x) + \nabla f(x)^\top (y - x),$$

  for all $x, y \in \mathrm{dom}(f)$.

- *Second order condition*
  Suppose f is twice differentiable at every point in its domain (so, in particular, $\mathrm{dom}(f)$ will need to be an open set when one applies this criterion). Then f is convex if and only if its Hessian $\nabla^2 f(x)$ is positive semidefinite at every $x \in \mathrm{dom}(f)$.

- *Restriction to a line*
  A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff for all $x \in \mathrm{dom}(f)$ and all $v \in \mathbb{R}^n$, the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(x + tv)$, with its domain defined to be $\mathrm{dom}(g) := \{t \ : \ x + tv \in \mathrm{dom}(f)\}$, is convex.

- *Epigraph is convex*
  A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph, defined as epi$(f) := \{(x,t) \ : \ x \in \mathrm{dom}(f), f(x) \le t\}$ is a convex subset of $\mathbb{R}^{n+1}$.

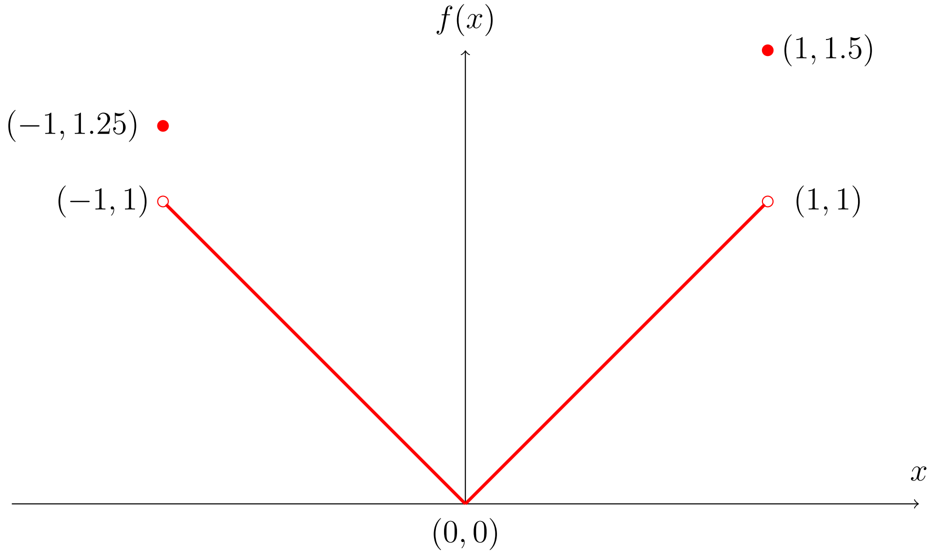

Figure 1: A graph of the function f in Problem 2(e). The function can be thought of as taking the value ∞ outside its domain, $\text{dom}(f) = [-1, 1]$. Note that the function jumps at -1 and at 1 in the sense that $\lim_{x\uparrow 1} f(x) = 1 \neq f(1) = 1.5$ and $\lim_{x\downarrow -1} f(x) = 1 \neq f(-1) = 1.25$. The existence of such jumps is emphasized by the open circles and the closed circles in the diagram. With the interpretation that the function equals ∞ outside the closed interval $[-1, 1]$ we would also have $\lim_{x\downarrow 1} f(x) = \infty \neq f(1) = 1.5$ and $\lim_{x\uparrow -1} f(x) = \infty \neq f(-1) = 1.25$.

(a) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, $x_1, x_2, \ldots, x_k \in \text{dom}(f)$, and $\theta_1, \theta_2, \ldots, \theta_k \geq 0$ with $\sum_{i=1}^{k} \theta_i = 1$, then we have

$$f(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \theta_2 f(x_2) + \cdots + \theta_k f(x_k).$$

This result is known as *Jensen's inequality* for discrete probability distributions.

(b) Find a condition on the symmetric matrix $A \in \mathbb{R}^{n\times n}$ in order to ensure that $f(x) = x^\top Ax$ is a convex function from $\mathbb{R}^n$ to $\mathbb{R}$.

(c) Show that the following operations preserve convexity:

i. **Nonnegative weighted sum:** If f and g are convex functions on $\mathbb{R}^n$ with domains $\text{dom}(f)$ and $\text{dom}(g)$ respectively, then $af + bg$ for $a, b \geq 0$ is a convex function on $\mathbb{R}^n$ with domain $\text{dom}(f) \cap \text{dom}(g)$, assuming $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$.

ii. **Pointwise maximum:** If f and g are convex functions on $\mathbb{R}^n$ with domains $\text{dom}(f)$ and $\text{dom}(g)$ respectively, then so is their pointwise maximum h, defined as

$$h(x) := \max\{f(x), g(x)\},$$

with domain $\text{dom}(f) \cap \text{dom}(g)$, assuming $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$.

(d) Show that $f : \mathbb{S}^n_{++} \to \mathbb{R}$ defined by $f(X) = \log\det(X)$ is concave. Note that we can think of f as a function on $\mathbb{S}^n$ (which is a vector space that contains $\mathbb{S}^n_{++}$), with $\operatorname{dom}(f) = \mathbb{S}^n_{++}$.

Hint: Use the "restriction to a line" condition to check for convexity.

(e) Consider the function $f : \mathbb{R} \to \mathbb{R}$ with $\operatorname{dom}(f) = [-1, 1]$, given by

$$f(x) = \begin{cases} 1.5 & \text{if } x = 1, \\ |x| & \text{if } -1 < x < 1, \\ 1.25 & \text{if } x = -1. \end{cases}$$

This is illustrated in Figure 1.

Show that f is a convex function.

3 Square-to-linear function

Consider the *square-to-linear* function

$$f(x, y) := \begin{cases} \frac{x^T x}{y} & \text{if } y > 0, \\ \infty & \text{otherwise,} \end{cases}$$

defined for $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $\text{dom}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \ : \ y > 0\}$.

(a) Show that $\text{dom}(f)$ is a convex subset of $\mathbb{R}^{n+1}$.

(b) Show that the Hessian of f at $(x, y) \in \text{dom}(f)$ is given by

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y^2 I_n & -yx \\ -yx^T & x^T x \end{bmatrix}.$$

(c) Show that $\nabla^2 f(x, y)$ is PSD at every $(x, y) \in \text{dom}(f)$ and thereby conclude that f is a convex function.