1 A simple constrained optimization problem

Consider the optimization problem

\[
\min_{x_1, x_2 \in \mathbb{R}} f(x_1, x_2)
\]

subject to

\[
2x_1 + x_2 \geq 1, \\
x_1 + 3x_2 \geq 1, \\
x_1 \geq 0, \ x_2 \geq 0.
\]

(a) Sketch the feasible set.

(b) For each of the following objective functions, give the optimal set or the optimal value.

i. \( f(x_1, x_2) = x_1 + x_2. \)

ii. \( f(x_1, x_2) = -x_1 - x_2. \)

iii. \( f(x_1, x_2) = x_1. \)

iv. \( f(x_1, x_2) = \max\{x_1, x_2\}. \)

v. \( f(x_1, x_2) = x_1^2 + 9x_2^2. \)
2 Convex conjugates

For a function $f : \mathbb{R}^n \to \mathbb{R}$, not necessarily a convex function, with a domain $\text{dom}(f)$, which we assume to be nonempty, but not necessarily a convex set, we can define its conjugate (also called its convex conjugate, Fenchel conjugate or Legendre-Fenchel conjugate), $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ via the rule

$$f^*(z) := \sup_{x \in \text{dom}(f)} (z^T x - f(x)).$$

Note that $f^*$ is an extended real valued function and does not take the value $-\infty$. Also note that it is convenient to treat $f$ also as an extended real valued function, taking the value $\infty$ outside $\text{dom}(f)$, and with this viewpoint we can also write

$$f^*(z) = \sup_{x \in \mathbb{R}^n} (z^T x - f(x)). \quad (1)$$

Note that, as an extended real valued function, $f$ also does not take the value $-\infty$.

(a) We will now find the conjugate of the convex function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := -\log x$, with $\text{dom}(f) = \mathbb{R}^{++}$ in a sequence of steps. (You can assume that the logarithm is to the natural base.)

i. Verify that the given function is convex.

ii. Show that $f^*(z) = \infty$ for $z \geq 0$.

iii. Next consider $z < 0$. Show that $\sup_{x > 0} (zx + \log x)$ is achieved at $x = \frac{1}{|z|}$, and thereby show that $f^*(z) = -1 - \log |z|$.

iv. Putting the previous parts together, determine the conjugate $f^*$ of the given function.

(b) Let $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^n$. Find the conjugate of the function $f : \mathbb{R}^n \to \mathbb{R}$, with $\text{dom}(f) = \mathbb{R}^n$, given by $f(x) := \|x\|$.

**Hint:** Your answer will involve the dual norm $\| \cdot \|^*$. 


3 Replacing containment by inequalities

Let $K \subseteq \mathbb{R}^n$. In the theory of convex sets and functions, the function

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise}, \end{cases}$$

is called the *indicator function* of $K$. Note that this terminology is not consistent with the one used in probability theory.

(a) Suppose $K$ is a nonempty convex subset of $\mathbb{R}^n$. Show that $I_K$ is a convex function with domain $K$.

(b) Suppose $K$ is a nonempty closed convex subset of $\mathbb{R}^n$. Let $I_K^*$ denote the conjugate of the indicator function $I_K$. Show that $I_K^*$ is a convex function, with $\text{dom}(I_K^*)$ being nonempty.

**Hint:** In fact, the conjugate $f^*$ of *any* function $f : \mathbb{R}^n \to \mathbb{R}$ (convex or not) with nonempty domain $\text{dom}(f)$ (convex or not) is either a convex function or everywhere equal to $\infty$. You may find it easier to show this more general result.

(c) Let $K$ be nonempty closed convex subset of $\mathbb{R}^n$. Because $\text{dom}(I_K^*)$ is nonempty, as established in the preceding part of the problem, we can take the conjugate of $I_K^*$, which we denote by $I_K^{**}$. Show that $I_K^{**} = I_K$.

**Remark:** The claim in this part of the problem will not be true if $K$ is a convex set that is not closed. In this case what will happen is that $I_K^{**} = I_K$, where $\overline{K}$ denotes the closure of $K$. To get some intuition for this you can work out, for yourself, the case where $K$ is the open interval $(0, 1)$ in $\mathbb{R}$. In fact, you can try to prove for yourself that, more generally, if $K \subseteq \mathbb{R}^n$ is any nonempty set, then $I_K^{**} = I_{\text{co}(K)}$, where $\text{co}(K)$ denotes the closed convex hull of $K$ (i.e. the closure of the convex hull of $K$).

**Remark:** Let $K$ be nonempty closed convex subset of $\mathbb{R}^n$. What we will have shown in this part of the problem is that

$$x \in K \iff I_K(x) = 0 \iff I_K(x) \leq 0 \iff I_K^*(x) \leq 0 \iff \sup_{z \in \mathbb{R}^n} (x^T z - I_K^*(z)) \leq 0 \iff x^T z \leq I_K^*(z) \text{ for all } z \in \mathbb{R}^n.$$ 

This way of expressing a containment constraint in terms of a family of linear constraints is what lies at the heart of duality in convex optimization, and we will explore this in more detail in the coming lectures.