EECS 127/227AT **Fall 2020**
Discussion 5

1 A simple constrained optimization problem

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} & f(x_1, x_2) \\ \text{subject to } & 2x_1 + x_2 \geq 1, \\ & x_1 + 3x_2 \geq 1, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

(a) Sketch the feasible set.

(b) For each of the following objective functions, give the optimal set or the optimal value.

i. $f(x_1, x_2) = x_1 + x_2$.
ii. $f(x_1, x_2) = -x_1 - x_2$.
iii. $f(x_1, x_2) = x_1$.
iv. $f(x_1, x_2) = \max\{x_1, x_2\}$.
v. $f(x_1, x_2) = x_1^2 + 9x_2^2$.

2 Convex conjugates

For a function $f : \mathbb{R}^n \to \mathbb{R}$, not necessarily a convex function, with a domain $\mathrm{dom}(f)$, which we assume to be nonempty, but not necessarily a convex set, we can define its *conjugate* (also called its *convex conjugate*, *Fenchel conjugate* or *Legendre-Fenchel conjugate*), $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ via the rule

$$f^*(z) := \sup_{x \in \mathrm{dom}(f)} \left(z^T x - f(x) \right).$$

Note that f^* is an extended real valued function and does not take the value $-\infty$. Also note that it is convenient to treat f also as an extended real valued function, taking the value ∞ outside $\mathrm{dom}(f)$, and with this viewpoint we can also write

$$f^*(z) = \sup_{x \in \mathbb{R}^n} \left(z^T x - f(x) \right). \tag{1}$$

Note that, as an extended real valued function, f also does not take the value $-\infty$.

(a) We will now find the conjugate of the convex function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := -\log x$, with $\mathrm{dom}(f) = \mathbb{R}_{++}$ in a sequence of steps. (You can assume that the logarithm is to the natural base.)

   i. Verify that the given function is convex.

   ii. Show that $f^*(z) = \infty$ for $z \geq 0$.

   iii. Next consider $z < 0$. Show that $\sup_{x>0} (zx + \log x)$ is achieved at $x = \frac{1}{|z|}$, and thereby show that $f^*(z) = -1 - \log |z|$.

   iv. Putting the previous parts together, determine the conjugate f^* of the given function.

(b) Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Find the conjugate of the function $f : \mathbb{R}^n \to \mathbb{R}$, with $\mathrm{dom}(f) = \mathbb{R}^n$, given by $f(x) := \|x\|$.

   Hint: Your answer will involve the dual norm $\|\cdot\|^*$.

3 Replacing containment by inequalities

Let $K \subseteq \mathbb{R}^n$. In the theory of convex sets and functions, the function

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise,} \end{cases}$$

is called the *indicator function* of K. Note that this terminology is not consistent with the one used in probability theory.

(a) Suppose K is a nonempty convex subset of $\mathbb{R}^n$. Show that I_K is a convex function with domain K.

(b) Suppose K is a nonempty closed convex subset of $\mathbb{R}^n$. Let I_K^* denote the conjugate of the indicator function I_K. Show that I_K^* is a convex function, with $\mathrm{dom}(I_K^*)$ being nonempty.

Hint: In fact, the conjugate f^* of *any* function $f : \mathbb{R}^n \to \mathbb{R}$ (convex or not) with nonempty domain $\mathrm{dom}(f)$ (convex or not) is either a convex function or everywhere equal to ∞. You may find it easier to show this more general result.

(c) Let K be nonempty closed convex subset of $\mathbb{R}^n$. Because $\mathrm{dom}(I_K^*)$ is nonempty, as established in the preceding part of the problem, we can take the conjugate of I_K^*, which we denote by I_K^{**}. Show that $I_K^{**} = I_K$.

Remark: The claim in this part of the problem will not be true if K is a convex set that is not closed. In this case what will happen is that $I_K^{**} = I_{\bar{K}}$, where $\bar{K}$ denotes the closure of K. To get some intuition for this you can work out, for yourself, the case where K is the open interval $(0, 1)$ in $\mathbb{R}$. In fact, you can try to prove for yourself that, more generally, if $K \subset \mathbb{R}^n$ is any nonempty set, then $I_K^{**} = I_{\bar{\text{co}}(K)}$, where $\bar{\text{co}}(K)$ denotes the closed convex hull of K (i.e. the closure of the convex hull of K).

Remark: Let K be nonempty closed convex subset of $\mathbb{R}^n$. What we will have shown in this part of the problem is that

$$\begin{aligned} x \in K \quad &\Leftrightarrow \quad I_K(x) = 0 \\ &\Leftrightarrow \quad I_K(x) \le 0 \\ &\Leftrightarrow \quad I_K^{**}(x) \le 0 \\ &\Leftrightarrow \quad \sup_{z \in \mathbb{R}^n} \left(x^T z - I_K^*(z) \right) \le 0 \\ &\Leftrightarrow \quad x^T z \le I_K^*(z) \text{ for all } z \in \mathbb{R}^n. \end{aligned}$$

This way of expressing a containment constraint in terms of a family of linear constraints is what lies at the heart of duality in convex optimization, and we will explore this in more detail in the coming lectures.