EECS 127/227AT Discussion 2 Slides

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Q1

Definition (Real Symmetric Matrix)

Matrix $A \in \mathbb{R}^{n \times n}$ such that $A = A^\mathsf{T}$.

Theorem (Spectral Theorem for Real Symmetric Matrices)

If A is symmetric, then

- *A has n orthogonal eigenvectors (that span $\mathbb{R}^n$)*
- *$A = U \Lambda U^\mathsf{T}$, where U's columns are the orthonormal eigenvectors of A, and Λ contains the corresponding eigenvectors (i.e. $Au_i = \lambda_i u_i$, where $u_i = U_{:,i}$ and $\lambda_i = \Lambda_{i,i}$).*

This is called the eigendecomposition of A.

Definition (Quadratic Function)

Function $\mathbb{R}^n \to \mathbb{R}$ that can be written as

$$f(x) = x^\mathsf{T} A x + b^\mathsf{T} x + c.$$

Without loss of generality A is symmetric (since otherwise write $A' = \frac{A + A^\mathsf{T}}{2}$).

Definition (Quadratic Form)

Function $\mathbb{R}^n \to \mathbb{R}$ with the form

$$f(x) = x^\mathsf{T} A x.$$

Quadratic function with $b = 0$, $c = 0$.

Definition (Positive Semidefinite Matrix)

A is positive semidefinite if $f(x) = x^\mathsf{T} A x \geq 0$ for all x.

Lemma

This definition is equivalent to all eigenvalues of A being ≥ 0.

Proof.

A is symmetric so we can write $A = U\Lambda U^T$, where U has orthonormal columns and Λ is diagonal. Then

$x^T Ax = \left(U^T x\right)^T \Lambda \left(U^T x\right) \geq 0$.

Proof that $x^T Ax \geq 0$ implies $\lambda_i \geq 0$: since $U^T U = I$, pick $x = Ue_i$, then $x^T Ax = \lambda_i \geq 0$.

Proof that $\lambda_i \geq 0$ implies $x^T Ax \geq 0$: Write $y = U^T x$, then $y^T \Lambda y = \sum_i \lambda_i y_i^2 \geq 0$, so $x^T Ax = x^T U\Lambda U^T x = y^T \Lambda y \geq 0$. □

Q2/3

Main idea: study linear subspaces associated with a matrix $A \in \mathbb{R}^{m \times n}$, where $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$.

Definition (Range)

Range or "column space" of A is the range of the function $f(x) = Ax$:

$$\text{range}(A) = \{Ax \,|\, x \in \mathbb{R}^n\} = \text{span}(a_1, \ldots, a_n) \subseteq \mathbb{R}^m.$$

Since span of vectors is a vector space, range(A) is a vector (sub)space. Also $\mathcal{R}(A)$.

Definition (Rank)

$\text{rank}(A) = \dim(\text{range}(A))$.

Definition (Null Space)

The null space or “kernel” of A is the space

$$\text{null}(A) = \{x \mid Ax = 0\} \subseteq \mathbb{R}^n.$$

This is also a vector (sub)space (closed under linear combinations). Also $\mathcal{N}(A)$.

Definition (Nullity)

$\text{nullity}(A) = \dim(\text{null}(A))$.

Theorem (Fundamental Theorem of Linear Algebra)

- $\text{range}(A)^{\perp} = \text{null}\left(A^{\mathsf{T}}\right)$
- $\text{null}(A)^{\perp} = \text{range}\left(A^{\mathsf{T}}\right)$

Corollary (Rank-Nullity Theorem)

$$\dim(\text{range}(A)) + \dim(\text{null}(A)) = n.$$

Recall n is the dimension of the domain of $A \in \mathbb{R}^{m \times n}$ (which takes vectors from $\mathbb{R}^n \to \mathbb{R}^m$).