EECS 127/227AT **Fall 2020**

Homework 0

Solutions for Problems 3 onwards in Homework 0 will be posted at 10 a.m. on Monday 8/31.
Please attempt the homework for self-evaluation before looking at the solutions.
If you have never used gradescope before, please make sure to submit your answers to Problems 1 and 2 on Homework 0 on Gradescope by 11.59 p.m. on Friday 9/4. This is to make sure that you can figure out how Gradescope works and konw to ask the GSIs for help if needed.
If you are already familiar with Gradescope, then submitting Problems 1 and 2 on Homework 0 on Gradescope is encouraged, but not essential.
The remaning problems in Homework 0 are for self-evaluation and are not to be submitted.
Self-graded Homework 0 is not to be submitted.

1. *Course setup*

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at **`https://inst.eecs.berkeley.edu/~eecs127/fa20/`** and familiarize yourself with the syllabus.

   (b) Register for the class Piazza at **`https://piazza.com/class/ke7o7uwalvo19`**

   (c) Register for the class Gradescope at **`https://www.gradescope.com`** using code **9RJR7Z**

   (d) When is homework due in general? When is the self-graded homework due in general? Where can you find the link to generate self grades? Where should homework be submitted? Where should self-graded homework be submitted?

   (e) How many homework drops do you get? Are there exceptions?

2. *Prerequisites and time zone*

   (a) What prerequisites have you taken?

   The prerequisites for this course are:

   - EECS 16A & 16B (Designing Information Devices and Systems I & II) **OR** MATH 54 (Linear Algebra & Differential Equations),
   - CS 70 (Discrete Mathematics & Probability Theory), and
   - MATH 53 (Multivariable Calculus).

   Please list which of these courses you have taken. If you have taken equivalent courses at a separate institution, please list them here. Indicate which of your courses corresponds to which of the prerequisites.

   If you are unsure of course material overlap, please refer to the EECS 16A, EECS 16B, and CS 70 websites:
   `https://www.eecs16a.org/`,
   `https://www.eecs16a.org/`,
   and `http://www.eecs70.org/`, respectively,
   and the MATH 53 textbook *Multivariable Calculus* by James Stewart.

   We will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it. Also, *there will be a review session on linear algebra and multivariate calculus on Friday Aug. 28*, time to be announced. Please participate in this if you feel it may help you (strongly encouraged).

   (b) If you have not yet done so, please fill out the Google form survey at
   `https://forms.gle/NdK8KA5cAAoH4rf5A`,
   particularly if you will not be in the U.S. Pacific time zone at any time during the semester.

3. *Solving systems of linear equations*

(a) I. Does the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 2x_2 + x_3 &= 3, \\ x_1 + 2x_2 + x_3 &= 2, \end{aligned} \tag{1}$$

have a solution?
You should be able to answer this question without actually finding a solution, if any.

II. Find all the solutions of the system of equations (1)

(b) I. Does the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 2x_2 + x_3 &= 2, \\ 2x_1 + 4x_2 + 6x_3 &= 2, \end{aligned} \tag{2}$$

have a solution?
You should be able to answer this question without actually finding a solution, if any.

II. Find all the solutions of the system of equations (2)

(c) I. Does the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 2x_2 + x_3 &= 2, \\ 4x_1 + 4x_2 + 4x_3 &= 1, \end{aligned} \tag{3}$$

have a solution?
You should be able to answer this question without actually finding a solution, if any.

II. Find all the solutions of the system of equations (3)

4. *Diagonalization of square matrices*

(a) Let $A := \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

I. Compute the eigenvalues of A (they will turn out to be real and distinct). Find nonzero eigenvectors of A for each eigenvalue.

II. Express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix.

(b) Let $A := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

I. Find the eigenvalues of A (they will turn out to be distinct complex numbers). Find nonzero eigenvectors for each eigenvalue (they will need to be vectors with complex entries).

II. Show that it is not possible to express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix if we require that P and Λ have real entries. However, it is possible to do this if we allow P and Λ to have complex entries.

(c) Let $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

I. Find the eigenvalues of A (it will turn out that there is only one eigenvalue, which will be of algebraic multiplicity 2 since it shows up as a double root of the characteristic polynomial of A). Find a nonzero eigenvector for each eigenvalue. (Please note that in this class the default assumption is that we are always talking about vectors and matrices with real entries, unless explicitly stated otherwise, so what is being asked is: find a nonzero eigenvector in $\mathbb{R}^2$ for each eigenvalue.)

II. Show that it is impossible to express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix. (Again, please note that, by default, what is being asked is to show that this is impossible, assuming that the matrices P and Λ have real entries.)

5. *Calculus*

(a) Let $f(x) := \sin x$, with domain $\mathbb{R}$. Let $g(x) := x^3$, with domain $\mathbb{R}$. Let

$$h(x) := g(f(x)),$$

denote the composition of g with f, with domain $\mathbb{R}$.

Find formulas for $h'(x)$ and $h''(x)$. Here, as usual, $h'(x)$ is alternate notation for $\frac{dh}{dx}(x)$, and $h''(x)$ is alternate notation for $\frac{d^2h}{dx^2}(x)$.

(b) Let $f(x) := \frac{e^x}{\cos x + x^2 - \frac{\pi^2}{4}}$, with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Find a formula for $f'(x)$.

6. *Multivariate calculus*

(a) The *log-sum-exp function* is the function lse : $\mathbb{R}^n \to \mathbb{R}$ given by

$$\mathrm{lse}(x) := \log\left(\sum_{i=1}^{n} e^{x_i}\right),$$

where $x := \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$. Here the logarithm is to the natural base. Note that the log-sum-exp function is well-defined for all $x \in \mathbb{R}^n$ because the argument of the logarithm will be strictly positive for all $x \in \mathbb{R}^n$, so we can take its domain to be $\mathbb{R}^n$.

Find a formula for the gradient of the log-sum-exp function.

Recall that the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ such that there is an open ball centered at x which is contained in the domain of f (namely, such that x is in the *interior* of the domain of f). is given by

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix},$$

assuming that all the indicated partial derivatives are well-defined. In this course the gradient will be defined to be a column vector, as above. (You will find some books where the gradient is defined to be a row vector.)

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $x \in \mathbb{R}^n$, and suppose that x is in the interior of the domain of f. The *Jacobian* of f at x is the $m \times n$ matrix of partial derivatives

$$Df(x) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix},$$

assuming that all the partial derivatives exist (if any of these partial derivatives does not exist, we say that the Jacobian of f is not well-defined at x). Here, for $1 \le j \le m$, the function $f_j : \mathbb{R}^n \to \mathbb{R}$ denotes the j-the component of the function f. Namely

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix},$$

or, if you like, in more detail

$$f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix},$$

where, as is customary, we write $f(x_1, \dots, x_n)$ for $f(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix})$.

I. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ is in the interior of the domain of f.
Show that the Jacobian of f at x is the transpose of the gradient of f at x.

II. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ with domain $\mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}$ with domain $\mathbb{R}^m$. Let $h(x) := g(f(x))$ denote the composition of g with f. Note that we have $h : \mathbb{R}^n \to \mathbb{R}$, with domain $\mathbb{R}^n$.
Show that for all $x \in \mathbb{R}^n$ we have

$$Dh(x) = Dg(f(x))Df(x),$$

assuming that all the desired partial derivatives exist. (Note that this is a matrix equation, since $Dh(x)$ is a row vector of length n, $Dg(f(x))$ is a row vector of length m and $Df(x)$ is an $m \times n$ matrix.)

7. *An algorithm*

Let $(x(n) \in \mathbb{R}^2, n \geq 0)$ be a sequence of vectors, starting from the initial vector $x(0) \in \mathbb{R}^2$, produced by the following algorithm

$$\begin{aligned} x_1(n+1) &= \frac{1}{3}x_2(n), \quad n \geq 0, \\ x_2(n+1) &= \begin{cases} 2x_1(n) & \text{if } n \text{ is even, } n \geq 0, \\ -x_1(n) + x_2(n) & \text{if } n \text{ is odd, } n \geq 0. \end{cases} \end{aligned}$$

Here $x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$ for $n \geq 0$, i.e. $x_1(n)$ and $x_2(n)$ is the notation we use for the first and the second coordinate respectively of the vector $x(n) \in \mathbb{R}^2$.

Show that $x_1^2(n) + x_2^2(n) \to 0$ as $n \to \infty$. (Namely, you are supposed to prove that this holds irrespective of the initial condition.)