EECS 127/227AT **Fall 2020**

Homework 1

Homework 1 is due on Gradescope by Friday 9/11 at 11.59 p.m.

1 Norms

Recall that for $x \in \mathbb{R}^n$ the ℓ_p norm for $1 \le p < \infty$ is defined as $||x||_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$, while the ℓ_∞ norm is defined as $\|x\|_\infty := \max_i |x_i|$.

(a) Show that for $x \in \mathbb{R}^n$ we have $||x||_\infty = \lim_{p\to\infty} \|x\|_p$.

(b) Show that for every $x \in \mathbb{R}^n$ we have

$$\frac{1}{\sqrt{n}}\|x\|_2 \overset{(1)}{\le} \|x\|_\infty \overset{(2)}{\le} \|x\|_2 \overset{(3)}{\le} \|x\|_1 \overset{(4)}{\le} \sqrt{n}\|x\|_2 \overset{(5)}{\le} n\|x\|_\infty.$$

Further show that every one of these inequalities is tight in that there is some nonzero $x \in \mathbb{R}^n$ for which that inequality is an equality.

(c) Show that for every nonzero vector $x \in \mathbb{R}^n$ we have

$$\text{card}(x) \ge \frac{\|x\|_1^2}{\|x\|_2^2},$$

where card(x) is the *cardinality* of the vector x, defined as the number of nonzero elements in x. Find all the nonzero vectors $x \in \mathbb{R}^n$ for which the lower bound is attained.

2 Bounds on the derivative of a polynomial

Consider the polynomial $p(x)$ in the real variable x, given by

$$p(x) := w_0 + w_1 x + \ldots + w_k x^k.$$

Here $w_0, \ldots, w_k$ are the coefficients of the polynomial, which are assumed to be real numbers. Assume that k, the degree of the polynomial, is at least 1. Let $v := \begin{bmatrix} w_1 & w_2 & \ldots & w_k \end{bmatrix}^T \in \mathbb{R}^k$.

(a) Using Hölder's inequality, show that for all $x \in [-1, 1]$ we have

$$\left| \frac{dp}{dx}(x) \right| \leq k \|v\|_1.$$

(b) Using the Cauchy-Schwarz inequality (which is a special case of Hölder's inequality), show that for all $x \in [-1, 1]$ we have

$$\left| \frac{dp}{dx}(x) \right| \leq k^{\frac{3}{2}} \|v\|_2.$$

(c) Using Hölder's inequality, show that for all $x \in [-1, 1]$ we have

$$\left| \frac{dp}{dx}(x) \right| \leq \frac{k(k+1)}{2} \|v\|_\infty.$$

3 Orthogonal projection onto a subspace

Let $\mathcal{S}$ denote the subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 := \begin{bmatrix} 2 & 2 & -1 & 0 \end{bmatrix}^T$ and $x_2 := \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$.

(a) What is the dimension of the subspace $\mathcal{S}$?

(b) Find a matrix A such that for every vector $x \in \mathbb{R}^4$ the vector Ax is the orthogonal projection of x onto the subspace $\mathcal{S}$.

4 Matrix Norms

A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called a matrix norm if the following properties hold:

(i) $f(A) \geq 0, \forall A \in \mathbb{R}^{m\times n}$, with $f(A) = 0$ iff $A = 0$.

(ii) $f(A + B) \leq f(A) + f(B), \forall A, B \in \mathbb{R}^{m\times n}$

(iii) $f(\alpha A) = |\alpha| f(A), \forall \alpha \in \mathbb{R}, A \in \mathbb{R}^{m\times n}$

These are just the properties that define a norm on $\mathbb{R}^{m\times n}$ when it is viewed as a vector space of dimension mn. (Note that the textbook writes $\mathbb{R}^{m\times n}$ as $\mathbb{R}^{m,n}$. Both forms of notation are widely used.)
One of the most frequently used matrix norms is the Frobenius norm

$$\|A\|_F := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

where a_{ij} denotes the entry at row i and column j of the matrix A. Just as in the case of $\|\cdot\|_p$ for vectors, we use the same notation $\|\cdot\|_F$ for the Frobenius norm irrespective of the dimensions of the underlying matrix.

(a) Prove that the Frobenius norm is indeed a matrix norm, i.e. that it satisfies the properties (i), (ii) and (iii).

(b) Prove that for $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}$, we have $\|AB\|_F \leq \|A\|_F \|B\|_F$.

(c) Prove that for a given matrix A and any orthogonal matrices P, Q of the appropriate dimensions we have $\|PAQ\|_F = \|A\|_F$.

5 Functions of a Matrix

Let $\lambda \in \mathbb{C}$ be an eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$.

(a) Assume $f : \mathbb{R} \mapsto \mathbb{R}$ is a polynomial of degree $m \in \mathbb{N}$, say

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

Show that $f(\lambda)$ is an eigenvalue of $f(A)$, where $f(A)$ is the matrix defined via

$$f(A) := a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m.$$

(b) Consider the Taylor series expansion of e^x around $x = 0$, given by

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} x^k,$$

and define the matrix exponential of a matrix $A \in \mathbb{R}^{n\times n}$ as

$$\exp(A) := I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k.$$

The infinite sum of matrices on the right hand side of this expression can be shown to be well-defined because we can find a constant $M > 0$ such that every entry of A^k is bounded in absolute value by M^k, and we have $\sum_{k=l}^{\infty} \frac{M^k}{k!} \to 0$ as $l \to \infty$.
Show that e^λ is an eigenvalue of the matrix $\exp(A)$.

(c) Let

$$C := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find $\exp(C)$.

(d) Note that $C = A + B$ where

$$A := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. \text{ and } B := \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find $\exp(A)$ and $\exp(B)$ and show that, even though $C = A + B$, we have

$$\exp(C) \neq \exp(A)\exp(B).$$

If you are wondering why, the reason for this is that $AB \neq BA$.