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EE16 A

OH Tu-Th 11-12 noon.

212 Cory.

Reminders : • HW1 Self Grades - today

• HW2 Released, due Friday 11:59pm.

• Start HW early

• New discussion schedule - website.

Today:

- Span
- Linear dependance/independance
- No unique solutions if cols. lin. dependant.
- Matrix-vector multiplication as state transformation. (Prob 5)
- Matrix-matrix multiplication.

Last time: Column perspective. Matrix-Vect mult.

$$\begin{array}{r} \\ \\ \text{Experiment} \rightarrow \end{array} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$ Linear combination.

sensitivity

Object → Experiment → Measurement

$\vec{x}$ $\qquad A\vec{x} = \vec{b}$

$\vec{a}_1 = \vec{0}$ → We don't have any information about x_1

SPAN: $\text{Span}\{\vec{a}_1, \vec{a}_2 \cdots, \vec{a}_n\} = \left\{ \sum_{i=1}^{n} c_i \vec{a}_i \;\middle|\; c_i \in \mathbb{R} \right\}$.

Belongs

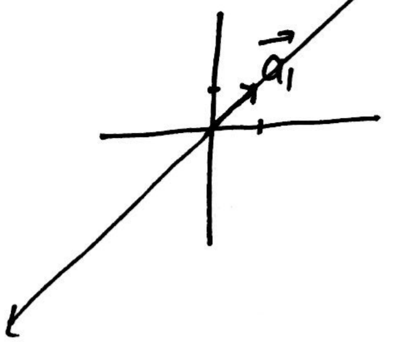
Real numbers

Set of all vectors that can be written as a linear combination of $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$.

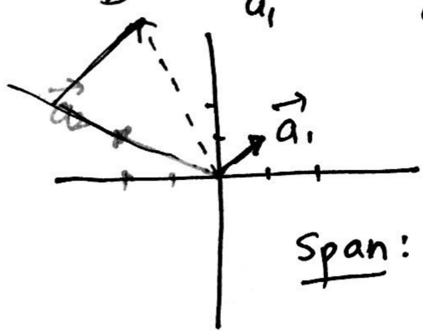
$\rightarrow c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n$

Example. $\text{Span}\left\{ \vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix} \right\}$.

$$= \left\{ c\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$



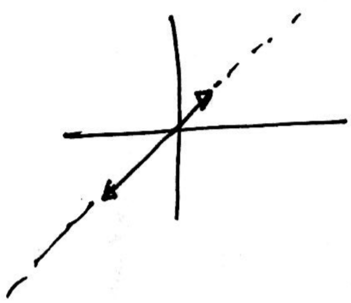
$$\text{Span}\left\{ \underset{\vec{a}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{a}_2}{\begin{bmatrix}-2\\1\end{bmatrix}} \right\}$$



Span: Entire 2D plane

$\mathbb{R}^2$

$$\text{Span}\left\{ \left[\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}-2\\-2\end{pmatrix} \right] \right\}$$



$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\-2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$\cancel{\begin{bmatrix}1 & -2\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}}$$

$$\begin{bmatrix}1 & -2\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}.$$

$$1c_1 - 2c_2 = b_1$$
$$1c_1 - 2c_2 = b_2.$$

$A\vec{x} = \vec{b}$ Solutions

Does $\vec{b}$ belong to the span(col. of A)?

$A\vec{x} = \vec{0}$ Homogenous solution.

$A\vec{x} = \vec{b}$ Particular solution.

Linear dependance:

Set of vectors $\vec{a}_1, \vec{a}_2 \dots \vec{a}_n$ are linearly dependant if one of the vectors can be written as a linear combination of the rest of the vectors.

There exists j such that

$$\vec{a}_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \vec{a}_i \Rightarrow \vec{0} = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \vec{a}_i - \vec{a}_j = \sum_{i=1}^{n} c_i \vec{a}_i$$

Defi… $c_j =$ …

$\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{a}_2 = \begin{bmatrix}-2\\-2\end{bmatrix}$ $\qquad \vec{a}_2 = -2\begin{bmatrix}1\\1\end{bmatrix}.$

$\vec{a}_1, \begin{bmatrix}1\\0\end{bmatrix}, \vec{a}_2 \begin{bmatrix}2\\0\end{bmatrix}.$ $\qquad \vec{a}_2 = 2 \cdot \begin{bmatrix}1\\0\end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{a}_2 = \begin{bmatrix}2\\1\end{bmatrix}, \vec{a}_3 = \begin{bmatrix}3\\1\end{bmatrix}.$ $\rightarrow$ Linearly dependant.

$$\vec{a}_3 = c_1 \cdot \vec{a}_1 + c_2 \vec{a}_2$$

Def 2: A set of vectors $\{\vec{a}_1, \dots \vec{a}_n\}$ is linearly dependant if there exist constants c_i, $\boxed{\text{not all zero}}$ such that $\sum_{i=1}^{n} c_i \vec{a}_i = \vec{0}$

Linear independence $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$. is lin. indep.

if they are not linearly dep

$\boxed{\text{If}}$. $c_1\vec{a}_1 + c_2\vec{a}_2 \dots + c_n\vec{a}_n = \vec{0}$, then $c_1 = c_2 = \dots = c_n = 0$.

Encoding for communication.

Transpose:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{x}^{T} = [x_1 \ x_2 \ \cdots \ x_n]$$

$$[x_1 \ x_2 \ \cdots \ x_n]^T \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Row

Alice $\rightarrow$ Bob.

$$\vec{a_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}. \qquad \vec{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

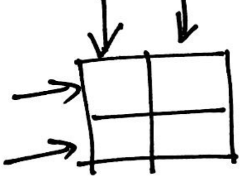
$$\beta_1 a_{11} + \beta_2 a_{21} + \beta_3 a_{31} = \underbrace{[\beta_1 \ \beta_2 \ \beta_3]} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \vec{\beta}^T \vec{a_1}$$

$$[\beta_4 \ \beta_5 \ \beta_6]$$

Why study lin dep?

$$\vec{x} \longrightarrow E \longrightarrow \vec{b}$$

$$A\vec{x} = \vec{b}$$



Thm: If the columns of A are linearly dependent, then $A\vec{x} = \vec{b}$ does not have a unique solution.

Proof : There exists ($\exists$) j such that

$$\vec{a_j} = \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \vec{a_i} \quad (*) \qquad A = \begin{bmatrix} | & | & & | \\ \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \\ | & | & & | \end{bmatrix}$$

LHS

$$A\vec{x} = \vec{a_j}$$

$$A\vec{x} = \vec{b}.$$

LHS

$$\vec{a_j} = A\cdot\vec{x}$$

$$\vec{z} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{j-1} \\ 0 \\ c_{j+1} \\ \vdots \\ c_n \end{bmatrix} \qquad A\vec{z} = c_1\cdot\vec{a_1} + c_2\vec{a_2} + \cdots c_n\vec{a_n}$$

$$= \vec{a_j} \qquad \text{by } (*).$$

$$\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th position} \qquad A\vec{y} = A\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \vec{a_j}$$

$$\vec{z} - \vec{y} \neq \vec{0}$$

$$A\vec{z} - A\vec{y} = \vec{0}$$

$$A(\underbrace{\vec{z}-\vec{y}}_{\text{non-zero vector.}}) = \vec{0}$$

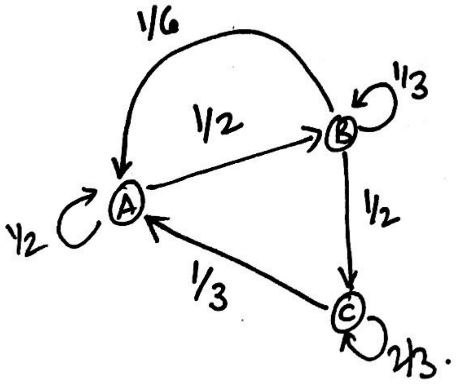
Say $\vec{w}$ is a solution $\quad A\vec{w} = \vec{b}$

$$A\vec{w} + A(\vec{z}-\vec{y}) = A(\underbrace{\vec{w} + (\vec{z}-\vec{y})}_{\neq \vec{w}}) = A\vec{w} = \vec{b}$$

<u>Thm:</u> If $A\vec{x} = \vec{b}$ has an infinite number of solutions, then the columns of A are linearly dependant.

$$\left.\begin{aligned} A\vec{x} &= \vec{b} \\ A\vec{y} &= \vec{b} \end{aligned}\right\} \quad A(\vec{x} - \vec{y}) = \vec{0}$$

<u>Matrix - vector product</u> : <u>Transformation of states.</u>



$$S_A(2) = \frac{1}{2} S_A(1) + \frac{1}{6} S_B(1) + \frac{1}{3} S_C(1)$$

$$S_B(2) = \frac{1}{2} S_A(1) + \frac{1}{3} S_B(1) + 0 \cdot S_C(1)$$

$$S_C(2) = 0 \cdot S_A(1) + \frac{1}{2} \cdot S_B(1) + \frac{2}{3} S_C(1).$$