EE16A
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OH Today 11-12 noon.
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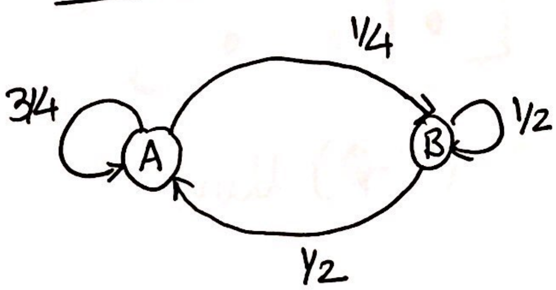
Reminders
HW due Friday.
Practice problems for midterm.
Labs on Monday

Today:

- Eigenvalues, Eigenspaces, Eigenvectors.
- $\lambda = 1$ steady state.
  $\lambda < 1$ transient
  $\lambda > 1$ exponentially growing, unstable.
- Change of basis.



$$Q = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

If λ is an eigenvalue.

$$Q - \lambda I = \begin{bmatrix} 3/4 - \lambda & 1/2 \\ 1/4 & 1/2 - \lambda \end{bmatrix}$$

Eigenspace corresponding to eigenvalue λ is the Nullspace $(Q - \lambda I)$.

Set of Eigenvector : Basis for the eigenspace.

$$\det(Q-\lambda I) = \left(\tfrac{3}{4}-\lambda\right)\left(\tfrac{1}{2}-\lambda\right) - \tfrac{1}{2}\cdot\tfrac{1}{4}.$$

$$= \frac{3}{8} + \lambda^2 - \frac{5}{4}\lambda - \frac{1}{8} = \lambda^2 - \frac{5}{4}\lambda + \frac{1}{4}$$

$$= (\lambda-1)\left(\lambda-\tfrac{1}{4}\right).$$

$\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$

$$(Q-I) = \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-x_1 + 2x_2 = 0$

$2x_2 = x_1$

eigenvector.

$$\text{Null}(Q-I) = \underbrace{\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}}_{\text{eigenspace}}$$

$Q\vec{v} = \lambda\vec{v}$

any $\vec{v}$ in this span

$$= \text{Span}\left\{\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}$$

$\lambda_2 = \frac{1}{4}$ $\qquad Q - \frac{1}{4}I = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{bmatrix}.$

$$\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$x_1 + x_2 = 0$

$$NS = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Are these a basis for $\mathbb{R}^2$

$\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

1. Lin indep.
2. $\begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & a-2b \end{array}\right]$$

$$y = \frac{-(a-2b)}{3}$$

$x + y = b \qquad x = b - y$

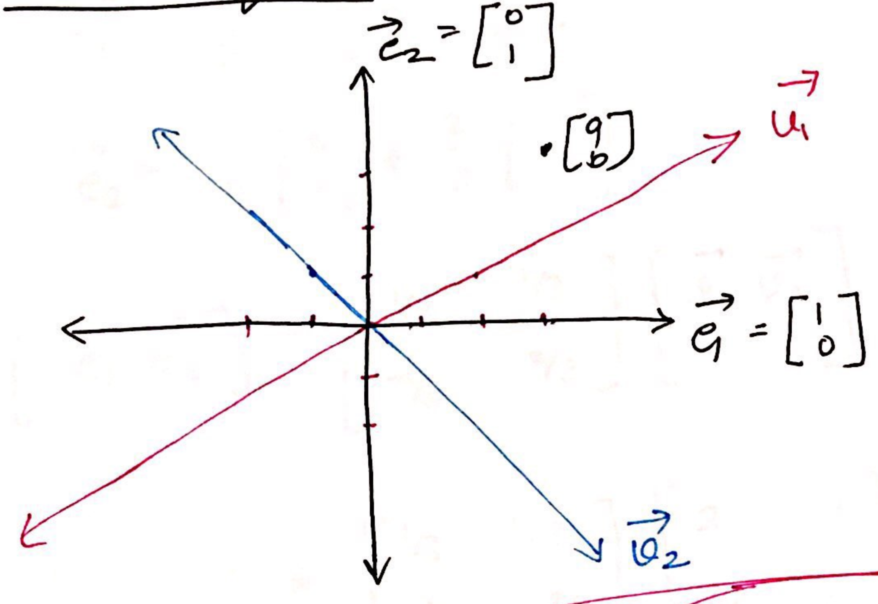
$$= b + \frac{(a-2b)}{3} = \frac{a+b}{3}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a+b}{3}\right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{-(a-2b)}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Change of Basis

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Coordinates.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left(\frac{-1}{3}\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

Basis $\vec{v}_1, \vec{v}_2$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Basis $\vec{e}_1, \vec{e}_2$

We know $\vec{v} = \begin{bmatrix} a' \\ b' \end{bmatrix}$ is a point in $\vec{v}_1, \vec{v}_2$

What was $\begin{bmatrix} a \\ b \end{bmatrix}$ in our standard basis?

$$\begin{bmatrix} a \\ b \end{bmatrix} = \vec{v} = a'\vec{v}_1 + b'\vec{v}_2 = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{V}\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = V^{-1}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{e_1} = \frac{1}{3}\overset{\vec{v_1}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \left(-\frac{1}{3}\right)\overset{\vec{v_2}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}.$$

$$\vec{e_2} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}}_{V^{-1}} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{V}$$

$$Q \cdot \vec{v_1} = \lambda_1 \cdot \vec{v_1} = \vec{v_1}$$

$$Q \cdot \vec{v_2} = \lambda_2 \vec{v_2} = \frac{1}{4} \cdot \vec{v_2}$$

$$\vec{x}(t) = \begin{bmatrix} a \\ b \end{bmatrix} \qquad Q \cdot \vec{x}(t) = \vec{x}(t+1).$$

$$= a' \vec{v_1} + b' \vec{v_2}$$

Let a', b' be the coordinates of $\vec{x}(t)$ in $\vec{v_1}, \vec{v_2}$

$$Q\vec{x}(t) = Q\left(a'\vec{v_1} + b'\vec{v_2}\right)$$

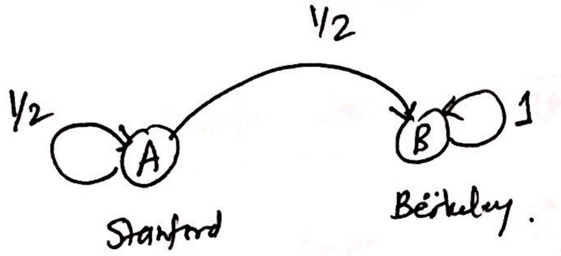
$$= a' \cdot Q \cdot \vec{v_1} + b' \cdot Q \cdot \vec{v_2}$$

$$= a' \vec{v_1} + b' \cdot \frac{1}{4} \cdot \vec{v_2}$$

$$Q^2 \cdot \vec{x}(t) = Q\left(a'\vec{v_1} + b' \frac{1}{4} \cdot \vec{v_2}\right)$$

$$= a' \cdot \vec{v_1} + b' \left(\frac{1}{4}\right)^2 \cdot \vec{v_2}$$

$$Q^m \cdot \vec{x}(t) = \underbrace{a' \vec{u_1}}_{\text{Steady state}} + \underbrace{b' \left(\tfrac{1}{4}\right)^m \cdot \vec{v_2}}_{\text{transient}}$$



$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$Q \cdot \vec{v_1} = \vec{v_1}$$

$$Q \cdot \vec{v_2} = \frac{1}{2} \vec{v_2}$$

$\longrightarrow \lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}$

$$Q - \frac{1}{2} I = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

e.space $\lambda_1 = 1/2 =$ Span$\left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v_2}} \right\}$

e.space. $\lambda = 1 =$ span$\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v_1}} \right\}$

$$Q\left(a'\vec{v_1} + b'\vec{v_2}\right) = a'\vec{v_1} + b' \cdot \frac{1}{2} \cdot \vec{v_2}$$

$$= Q \cdot a'\vec{v_1} + Q \cdot b\vec{u_2}$$

$$Q^m\left(a'\vec{v_1} + b'\vec{v_2}\right) = a' \cdot \vec{v_1} + \left(\frac{1}{2}\right)^m \cdot b\,\vec{v_2}$$

$$= a' \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right)^m \cdot b \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q(a\vec{e_1} + b\vec{e_2}) = aQ\vec{e_1} + b \cdot Q\vec{e_2}$$

Thm: For a 2x2 matrix, the eigenvectors corresponding to distinct eigenvalues form a basis $\mathbb{R}^2$

Given:

$$\boxed{A\vec{v_1} = \lambda_1 \vec{v_1}} \quad (1)$$

$$A\vec{v_2} = \lambda_2 \vec{v_2} \quad (2)$$

$\lambda_1 \neq \lambda_2$.

Want to show: $\vec{v_1}, \vec{v_2}$ are linearly independent

If possible let $\boxed{\vec{v_1} = \alpha \cdot \vec{v_2}}$ $(*)$ i.e. if possible, let $\vec{v_1}, \vec{v_2}$ be linearly dependent.

$\alpha \neq 0$

$$A\vec{v_1} = A(\alpha \vec{v_2}) = \alpha \cdot \lambda_2 \cdot \vec{v_2} \text{ from } (2).$$

$$(1) \Rightarrow A \cdot \vec{v_1} = \lambda_1 \vec{v_1} = \lambda_1(\alpha \cdot \vec{v_2})$$

from $(*)$

$$\alpha \cdot \lambda_2 \cdot \vec{v_2} = \lambda_1 \cdot \alpha \cdot \vec{v_2}$$

$\therefore$ Not possible, since $\lambda_1 \neq \lambda_2$.

Contradiction $\Rightarrow$ $\vec{v_1}, \vec{v_2}$ are lin. indep.