Module 3

Gireeja Ranade

Today

- Election Day.
- Locationing Systems
- Inner-products / Dot products.
- Correlation
- Trilateration.

Module 1: Modeling – How to think about the physical world.

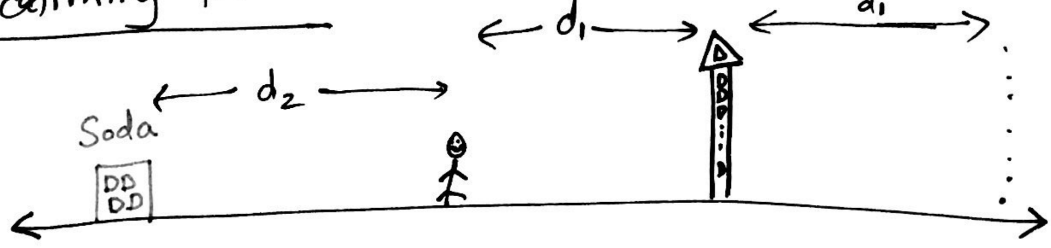
Module 2: How can we extract information from the physical world.
How do we design to achieve a goal.

Module 3: Process the information and "learn" from it.

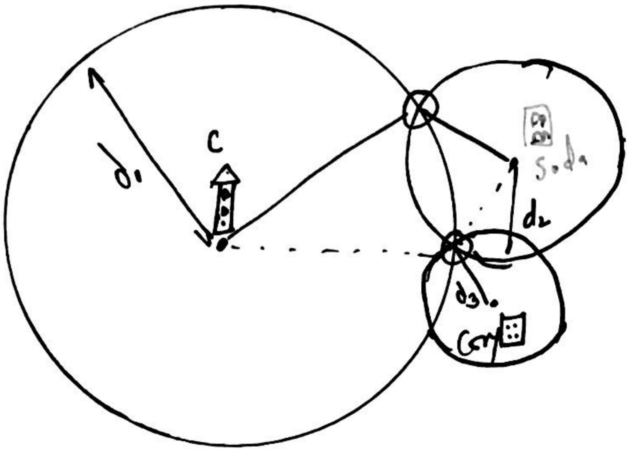
Lab: Locationing GPS →
(G ← global.)

APS
(A ← acoustic)

1D Locationing Problem.



2D locationing system



How do we do this?

1. Figure out the distance to the landmarks. → Correlation
2. Combine these distances to understand our location. → Trilateration
3. Deal with noise. → Least Squares algorithm.

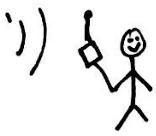
---

1. How to figure out distance.

Synchronized clock between Tx and Rx

Satellite.

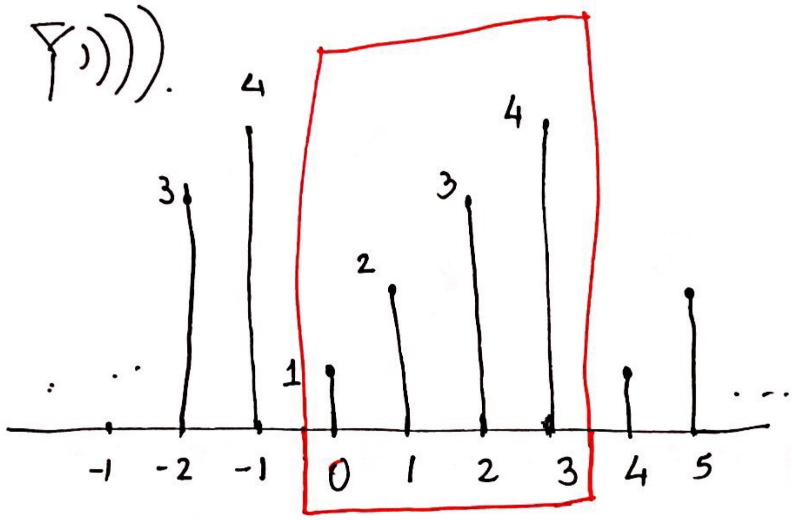
Send out signals.



Receivers receives with some delay

EM: 3×10^8 m/s

Sound: ~340 m/s.



Signal.

Represent: $x[n]$

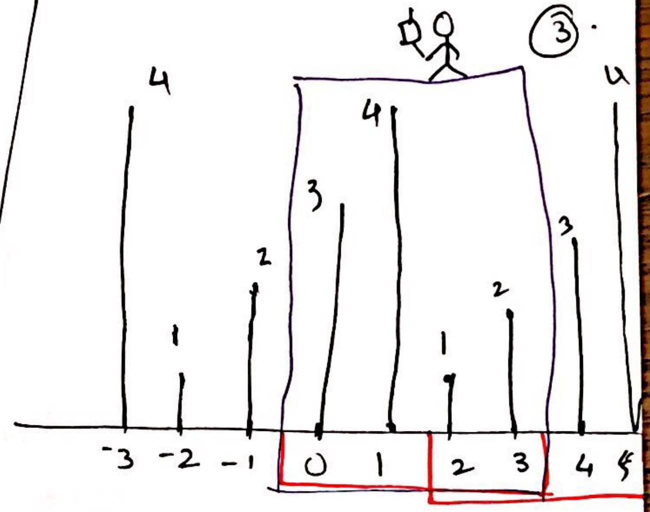
$x[0]=1$, $x[1]=2$, $x[2]=3$, $x[3]=4$

$x[n]=1$ if $n=4k$

$x[n]=2$ if $n=4k+1$

$x[n]=3$ if $n=4k+2$.

$x[n]=4$ if $n=4k+3$.



Receive: $y[n]$.

$y[n]=1$ if $n=4k+2$

$y[n]=2$ if $n=4k+3$.

$y[n]=3$ if $n=4k$

$y[n]=4$ if $n=4k+1$.

Transmitted signal: $s[n]$

Period N

$\{0, 1, \dots, N-1\}$ shifts.

$\boxed{\text{Delay: } d.}$??? ← Unknown.

$$\vec{s} = \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix}$$

$\vec{s}_k$ = shifted version of $\vec{s}$ by k.

$s[0]$ is at position k.

Received signal.

$r[n]$

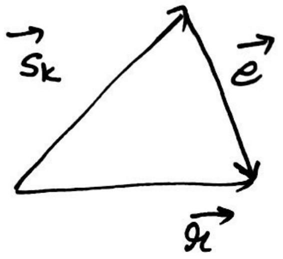
$$\vec{r} = \begin{bmatrix} r[0] \\ r[1] \\ \vdots \\ r[N-1] \end{bmatrix} = \begin{bmatrix} s[N-d] \\ \vdots \\ s[N-1] \\ s[0] \\ \vdots \\ s[N-d-1] \end{bmatrix}$$

← dth position

$\vec{s}_d$

Find d

Compare



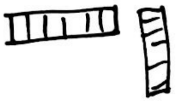
$$\vec{e} = \vec{r} - \vec{s}_k$$

smaller $\vec{e}$

$\rightarrow$ closer $\vec{s}_k$ is to $\vec{r}$

Reminder: Dot product / Inner product $\qquad \vec{x}, \vec{y} \in \mathbb{R}^n$

$$\vec{x}^T \vec{y} = \underset{1\times 1}{\square} \text{ scalar.} = \vec{y}^T \vec{x}$$



$$\vec{x}^T \vec{y} = \vec{x} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x[i] \cdot y[i]$$

$$\vec{s} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \end{bmatrix} = \vec{s}_0$$

$$\vec{r} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{s}_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{s}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{s}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{s}_0^{\,T} \cdot \vec{r} = 3 + 8 + 3 + 8 = \cancel{24}\ 22$$

$$\vec{s}_1^{\,T} \cdot \vec{r} = 12 + 4 + 2 + 6 = 24$$

$$\boxed{\vec{s}_2^{\,T} \cdot \vec{r} = 3^2 + 4^2 + 1^2 + 2^2 = 30}$$

$$\vec{s}_3^{\,T} \cdot \vec{r} = 6 + 12 + 4 + 2 = 24$$

Cross-correlation of $\vec{x}$ and $\vec{y}$ (Periodic with period N)

"Shifted dot product"

$$\mathrm{corr}_N(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i] \cdot y[i-k].$$

correlation sample k.

$$\mathrm{corr}_4(\vec{r}, \vec{s})[0] = \sum_{i=0}^{3} r[i]\, s[i] = 3 + 8 + 3 + 8 = 22$$

$$\mathrm{corr}_4(\vec{r}, \vec{s})[1] = \sum_{i=0}^{3} r[i] \cdot s[i-1] = r[0] \cdot \overset{s[3]}{s[-1]} + r[1]\, s[0] + r[2] \cdot s[1] + r[3] \cdot s[2]$$

$$= 12 + 4 + 2 + 6 = 24$$

$$\begin{bmatrix} s[0] & s[1] & s[2] & s[3] \\ s[3] & s[0] & s[1] & s[2] \\ s[2] & s[3] & s[0] & s[1] \\ s[1] & s[2] & s[0] & s[0] \end{bmatrix} \begin{bmatrix} r[0] \\ r[1] \\ r[2] \\ r[3] \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Circulant matrix

Correlation

$\vec{r}, \ \vec{s_k}$

$\vec{e} = \vec{r} - \vec{s_k}$.

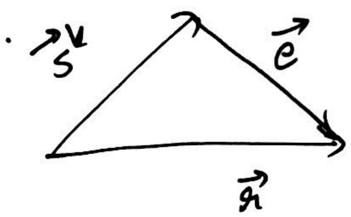
Minimize length of $\vec{e}$

Norm: $\|\vec{e}\|$ $\qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$= \sqrt{e_1^2 + e_2^2}$

$\|\vec{e}\|^2 = e_1^2 + e_2^2$

$= \vec{e}^T \cdot \vec{e} = \langle \vec{e}, \vec{e} \rangle$.



Length of $\vec{e}$ squared

Find k to minimize

$= \|\vec{e}\|^2$

$= \|\vec{r} - \vec{s_k}\|^2$

$= (\vec{r} - \vec{s_k})^T (\vec{r} - \vec{s_k})$

$= \vec{r}^T \cdot \vec{r} + \vec{s_k}^T \cdot \vec{s_k} - (\vec{s_k}^T \vec{r} + \vec{r}^T \cdot \vec{s_k})$

$= \|\vec{r}\|^2 + \underbrace{\|\vec{s_k}\|^2}_{= \|\vec{s}\|^2} - 2\langle \vec{s_k}, \vec{r} \rangle$