EE16A

Nov 13, 2018

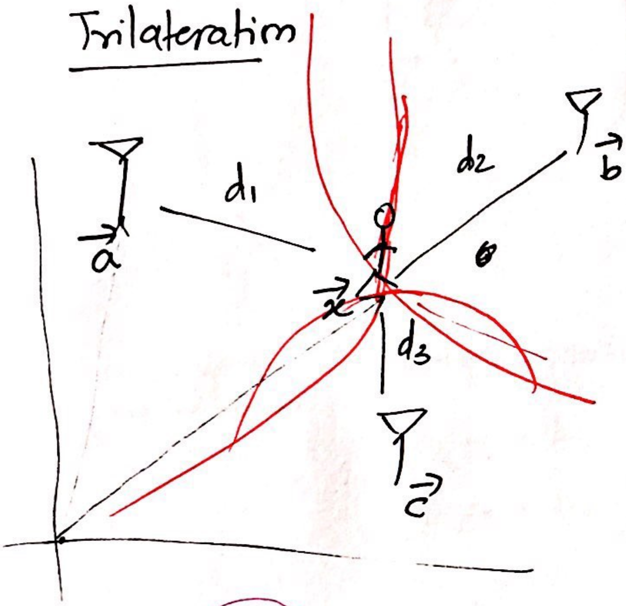
- Locationing
- Projections Revisited.
- Least Squares.

· MT2 Scores released.

· HW12 due Friday.

· HW13 last HW.

Trilateration



Known: $\vec{a}, \vec{b}, \vec{c}$

Unknown: $\vec{x}$

① $\|\vec{x} - \vec{a}\|^2 = d_1^2$

② $\|\vec{x} - \vec{b}\|^2 = d_2^2$

③ $\|\vec{x} - \vec{c}\|^2 = d_3^2$.

① $\|\vec{x}\|^2 + \|\vec{a}\|^2 - 2\langle \vec{x}, \vec{a} \rangle = d_1^2$

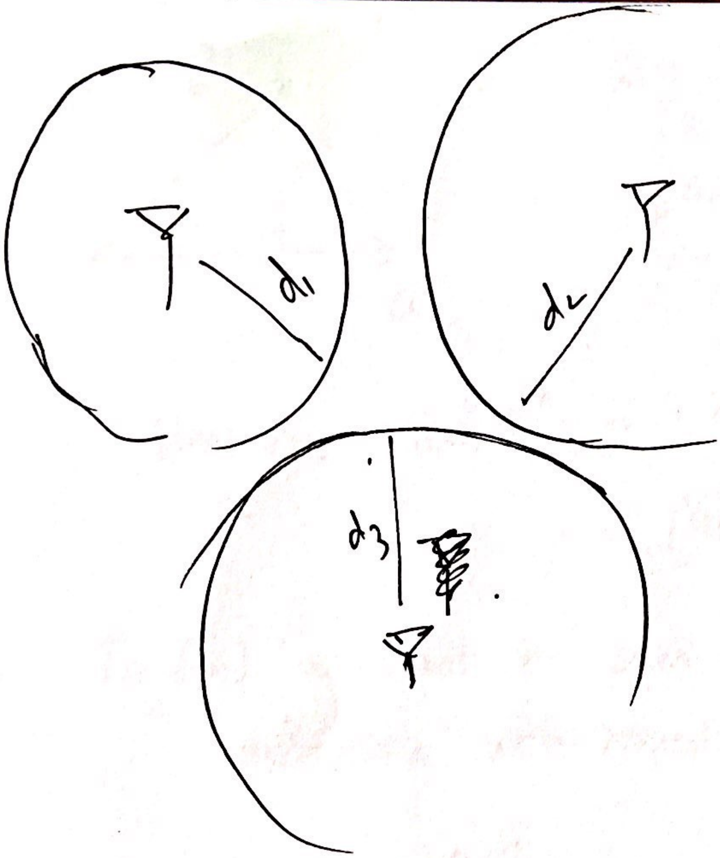
② $\|\vec{x}\|^2 + \|\vec{b}\|^2 - 2\langle \vec{x}, \vec{b} \rangle = d_2^2$

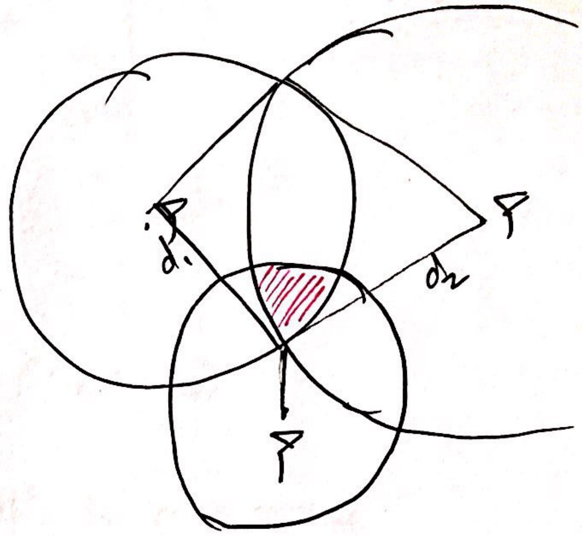
② - ① $\Rightarrow$ $\|\vec{b}\|^2 - \|\vec{a}\|^2 + 2\big(\overbrace{\langle \vec{x}, \vec{a} \rangle - \langle \vec{x}, \vec{b} \rangle}^{\langle \vec{x}, \vec{b} - \vec{a} \rangle}\big) = d_2^2 - d_1^2$

③ - ① $\Rightarrow$

$$2 \begin{bmatrix} (\vec{a} - \vec{b})^T \\ (\vec{a} - \vec{c})^T \end{bmatrix} \vec{x} = \begin{bmatrix} d_2^2 - d_1^2 + \|\vec{a}\|^2 - \|\vec{b}\|^2 \\ \text{---}\ \ ''\ \ \text{---} \end{bmatrix}$$

c's





Example:

Simple problem:

$a_1 x = b_1$ ①

$a_2 x = b_2$ ②

x is unknown, scalar.

Guess:

① $\Rightarrow$ $x = b_1/a_1$

② $\Rightarrow$ $x = b_2/a_2$

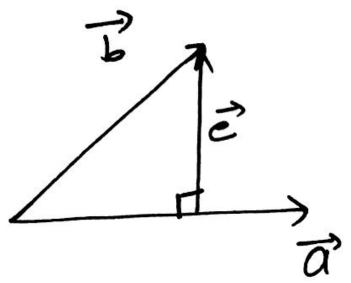
$$\hat{x} = \frac{\frac{b_1}{a_1} + \frac{b_2}{a_2}}{2}$$

$\uparrow$ estimate of x

$$\underset{\substack{A \\ 2\times 1}}{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} \underset{\substack{x \\ 1\times 1}}{x} = \underset{\substack{\vec{b} \\ 2\times 1.}}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

Question:

↳ Find $\vec{x}$ in the span of A that is closest to $\vec{b}$



Rewrite

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\text{Known}} x = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{Known}} + \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{\text{error}}$$

unknown

$$\underset{x}{\text{Minimize}} \ \|\vec{e}\|^2 = e_1^2 + e_2^2$$
$$= (a_1 x - b_1)^2 + (a_2 x - b_2)^2$$

To find argument x such that $\|\vec{e}\|^2$ is minimized, differentiate with respect to x, set to 0.

"Chain Rule"

$$0 = \frac{d}{dx} \|\vec{e}\|^2 = 2 \cdot a_1 \cdot (a_1 x - b_1) + 2 \cdot a_2 \cdot (a_2 x - b_2)$$

Solve for x:

$$0 = (a_1^2 + a_2^2) x + (-a_1 b_1 - a_2 b_2).$$

$$\Rightarrow \quad x = \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2} = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2}$$

"Projection"

$\therefore$ Projection of $\vec{b}$ onto $\vec{a}$ is the vector with minimum error.

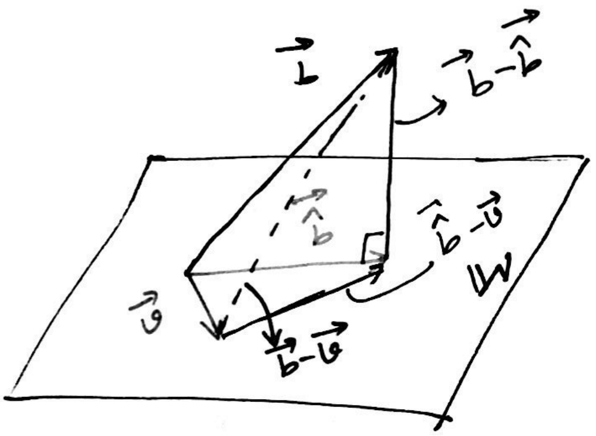
<u>Thm:</u> Let $\vec{b}$ be a vector and $\mathbb{W}$ be a subspace.

Let $\vec{\hat{b}}$ be a vector such that

- $\vec{e} = \vec{b} - \vec{\hat{b}}$ is orthogonal to $\mathbb{W}$.

$\vec{\hat{b}}$ is the orthogonal projection of $\vec{b}$ onto $\mathbb{W}$.

Then, $\vec{\hat{b}} \in \mathbb{W}$ is the vector that is closest to $\vec{b}$ in $\mathbb{W}$.



<u>Want:</u> For $\vec{v} \in \mathbb{W}$, $\vec{v} \neq \vec{\hat{b}}$.

$$\|\vec{b}-\vec{v}\|^2 > \|\vec{b}-\vec{\hat{b}}\|^2$$

<u>Proof:</u>

$$\vec{b}-\vec{v} = \vec{b}-\vec{\hat{b}} + \underbrace{\vec{\hat{b}}-\vec{v}}_{\text{belongs to } \mathbb{W}}$$

$\vec{b}-\vec{\hat{b}} \perp \mathbb{W}$

$\vec{b}-\vec{\hat{b}} \perp \vec{\hat{b}}-\vec{v}$

<u>Pythagoras</u> $\Rightarrow$ $\underbrace{\|\vec{\hat{b}}-\vec{v}\|^2}_{>0} + \|\vec{b}-\vec{\hat{b}}\|^2 = \|\vec{b}-\vec{v}\|^2$

$\Rightarrow \|\vec{b}-\vec{v}\|^2 > \|\vec{b}-\vec{\hat{b}}\|^2 \quad \rightarrow$ <u>DONE.</u>

Least Squares. Algorithm.

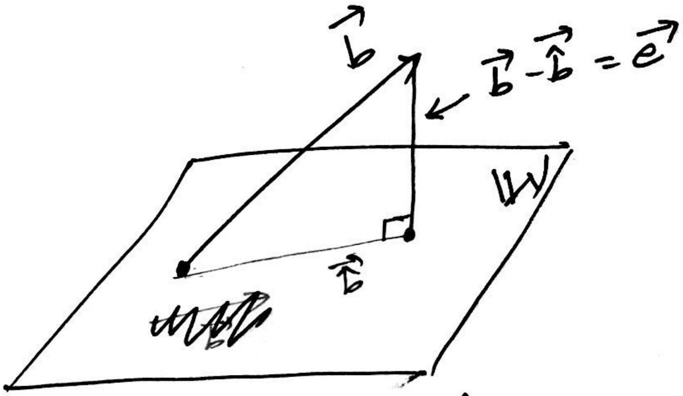
$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}}_{\text{measurement } m\times n \text{ matr}} \overbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}^{\text{unknowns}} \approx \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\text{data}}$$

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Overdetermined system

If $\vec{b} \in \text{span}(A)$ → great ✓

If $\vec{b} \notin \text{span}(A)$



W = set of linear combinations of $\vec{a}_1 \ldots \vec{a}_n$

$v \in W$

$\vec{v} = \alpha_1 \vec{a}_1 + \cdots \alpha_n \vec{a}_n$

$\langle \vec{b} - \hat{\vec{b}}, \vec{v} \rangle = 0$

$\vec{e}$ is orthogonal to W.

$$\langle \vec{b} - \hat{\vec{b}}, \vec{a}_1 \rangle = 0 \quad (1)$$

$$\langle \vec{b} - \hat{\vec{b}}, \vec{a}_2 \rangle = 0 \quad (2)$$

$$\vdots$$

$$\langle \vec{b} - \hat{\vec{b}}, \vec{a}_n \rangle = 0 \quad (n)$$

$$\langle \vec{b}, \vec{a}_1 \rangle = \langle \hat{\vec{b}}, \vec{a}_1 \rangle$$

$$\vec{a}_1^{T} \cdot \vec{b} = \vec{a}_1^{T} \cdot \hat{\vec{b}} \quad (1)$$

Scanned with CamScanner

$$\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \\ \vdots \\ - a_n^T - \end{bmatrix} \vec{b} = \begin{bmatrix} - a_1^T - \\ \vdots \\ - \vec{a}_n^T - \end{bmatrix} \vec{\hat{b}}$$

$$A^T \vec{b} = A^T \vec{\hat{b}}$$

$$\vec{\hat{b}} \in \text{Span}(A) \implies \vec{\hat{b}} = A \cdot \vec{x}$$

$$A^T \cdot \vec{b} = A^T \cdot (A \cdot \vec{x})$$

$$A^T \cdot \vec{b} = \underbrace{(A^T \cdot A)}_{\text{square matrix}} \cdot \vec{x}$$

$A^T A$: $n \times n$ sq. matrix

A^T: $n \times m$ $\quad$ A: $m \times n$

$$\boxed{\vec{x} = (A^T A)^{-1} \cdot A^T \vec{b}.}$$