EECS 16A

Sept 17, 2019.

Last time:

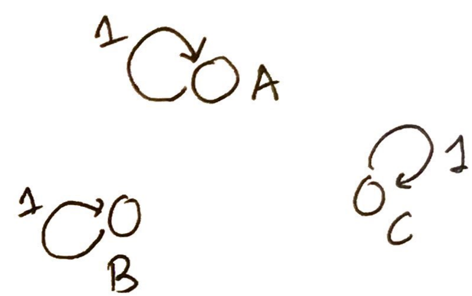
Matrix Multiplication as linear transformation.

$f(x) = 2x.$
$g(x) = \frac{1}{2}x$] inverses.

Today:

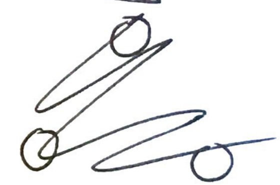
- Matrix inversion.
- A proof about inversion.
- Nullspaces.

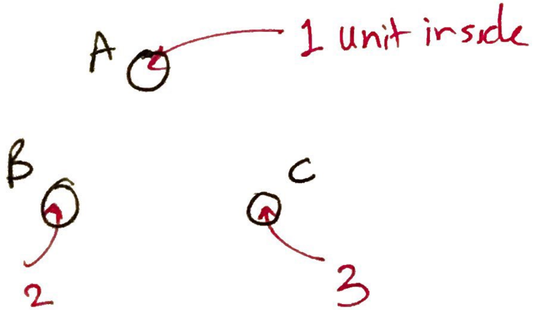
$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{x}(t+1) = Q \cdot \vec{x}(t)$$



"Identity" matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{x}(1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$





- Run Q once.

$$\boxed{\vec{x}(2) = Q \cdot \vec{x}(1)}$$
$$\vec{x}(3) = R \cdot \boxed{\vec{x}(2)}$$
$$= \boxed{R \cdot (Q} \cdot \vec{x}(1))$$

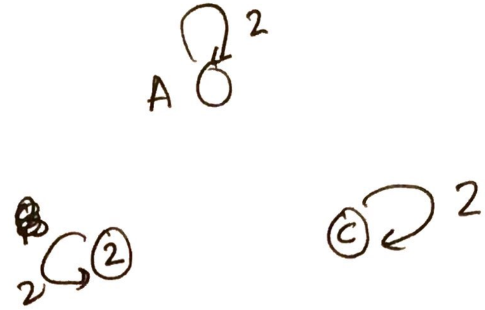
- Run R once

$$\vec{x}(2) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{x}(3) = \begin{bmatrix} 1/2 \\ 3/2 \\ 2 \end{bmatrix}$$

Run RQ.

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

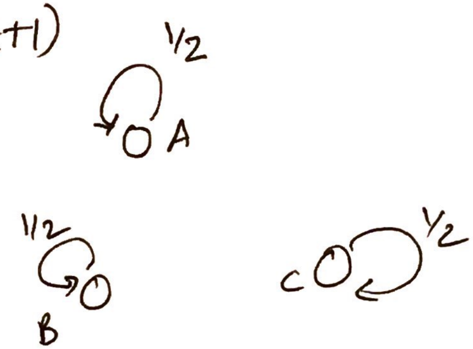


- If a pump generating water/energy etc.
- If a pump leaks.

→ "Non-conservative"

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

Recover: $\vec{x}(t)$ from $\vec{x}(t+1)$



$$R = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\vec{x}(t+1) = Q \cdot \vec{x}(t) \quad , \quad \boxed{\vec{x}(t) = \underbrace{R \cdot \vec{x}(t+1)}_{\text{subs.}}}.$$

$$= Q \cdot \boxed{R \cdot \vec{x}(t+1)}$$

$$= \boxed{Q \cdot R}\, \vec{x}(t+1)$$

$$\boxed{Q \cdot R = I}$$

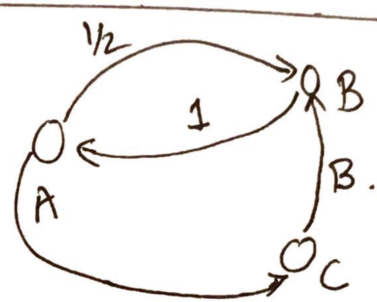
~~We know~~

$\cancel{A\vec{x} = \vec{b}}$

Definition: Matrix P is said to be the inverse of matrix Q (Both square) if $PQ = Q \cdot P = I$

Sometimes, we will denote $P = Q^{-1}$ "Q-inverse"

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$$



$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

Want: P such that $\vec{x}(t) = P \cdot \vec{x}(t+1)$

$$\cancel{\vec{x}(t) = P \cdot} \quad \vec{x}(t+1) = Q \cdot \vec{x}(t)$$
$$= Q \cdot P \cdot \vec{x}(t+1)$$

$$\Longrightarrow \quad Q \cdot P = I$$ Matrices instead of vectors.

Vectors

→ **We know:** $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\quad}_{\vec{P_1}} \quad \underbrace{\quad}_{\vec{P_2}} \quad \underbrace{\quad}_{\vec{P_3}}$$

$Q \cdot \vec{P_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $Q \cdot \vec{P_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $Q \cdot \vec{P_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Augmented matrix :

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Key observation: GE steps do not depend on the $\vec{b}$ for $A\vec{x} = \vec{b}$.

$$\xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \cdot 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_3(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{ccc} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \\ \vec{p_1} & \vec{p_2} & \vec{p_3} \end{array}$$

- Order: Thm: If $Q \cdot P = I$ and $R \cdot Q = I$, then $R = P$.

"Left inverse and the right inverse are the same"

→ Important because in general matrix multiplication does not commute.

Proof: Beginning:

$Q \cdot P = I$.

$R \cdot Q = I$.

End: $R = P$

$Q \cdot P = R \cdot Q$ → doesn't work.

$\rightarrow R \cdot (Q \cdot P) = R \cdot (I)$.

$\Rightarrow \underbrace{R \cdot Q}_{} \cdot P = R$.

$\Rightarrow I \cdot P = R$

$P = R$.

What we did not do:

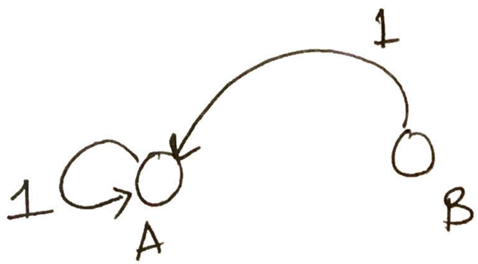
$(Q \cdot P) R = (I) \cdot R = R$

$Q \cdot P \cdot R = R$

$I \cdot R = R$

- $f(x) = 2x$, $g(x) = \frac{1}{2}x$
  $f(x) = 0 \cdot x$

e.g.



$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Cannot recover $\vec{x}(t)$ from $\vec{x}(t+1)$.

Inverses need not always exist.

Inversion $\longleftrightarrow$ Gaussian elimination $\longleftrightarrow$ Linear dependance + indep.
Unique solutions

Thm: If the columns of A are linearly dependant then $\Rightarrow$ Matrix A is not invertible.

$p \Rightarrow q$
raing clouds.

Another way to say the same thing
If matrix A is invertible, ie. A^{-1} exists
$\Longrightarrow$ the columns of A are linearly independant.

NOT q $\Rightarrow$ NOT p.
"contrapositive"

Proof: By def. of linear dependance.

End: A^{-1} does not exist

Know: there exist $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots c_n\vec{a}_n = 0$

where not all c_i's are ~~not~~ 0.

If possible: Let A^{-1} exist.

$$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0 \qquad \vec{c} \neq 0$$

$$A^{-1}(A \cdot \vec{c}) = A^{-1} \cdot 0$$ Multiply by A^{-1}

Use associative property.

$$\Rightarrow (A^{-1}A) \cdot \vec{c} = 0$$

$$\Rightarrow I \cdot \vec{c} = 0$$

$$\Rightarrow \vec{c} = 0$$

A contradiction ▨ QED