EECS 16A

Sept 19, 2019

Lecture 7

Last time:

- Matrix inversion. $\rightarrow$ Gaussian Elimination A^{-1}.
- Inverse doesn't always exist.
- **Thm:** If the columns of A are linearly dep., then A is not invertible.

Today:
- More inversion.
- Connect back to lab.
- Null spaces + Vector spaces.

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① **Thm:** If A is an invertible matrix, then $A\vec{x} = \vec{b}$ has a unique solution.

Proof:

Want to prove:

① There is atleast one solution. ✓

② All other solutions are equal to this.

Known: A^{-1} exists.

Consider: $\vec{x_0} = A^{-1} \cdot \vec{b}$

Check: $A \cdot \vec{x_0} = A(A^{-1} \cdot \vec{b})$

$= I \cdot \vec{b} = \vec{b}$

Let $\vec{x_1}$ be another solution.

$A\vec{x_1} = \vec{b}$

$A^{-1} \cdot A \cdot \vec{x_1} = A^{-1} \cdot \vec{b}$

$I \cdot \vec{x_1} = A^{-1} \cdot \vec{b}$

$\vec{x_1} = A^{-1} \cdot \vec{b} = \vec{x_0}$

QED

- If A is invertible $\iff$ $A\vec{x} = \vec{b}$ has a unique solution.
  $\iff$ A has linearly independent columns. $\Bigg]$ A is square

  $\implies$ A has a trivial nullspace.

$A\vec{x} = \vec{0}$ How many solutions?

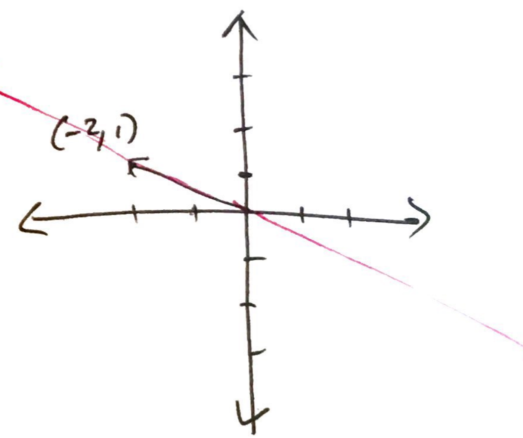
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$ Unique solution! $\vec{x_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow$ infinitely many! $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $A\vec{x} = \vec{0}$ has a solution, $\vec{x_0} \neq \vec{0}$

then $\alpha \vec{x_0}$ is also a solution! $\alpha \in \mathbb{R}$.

$\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ is a solution.



$A\vec{x} = \vec{0}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}$

Nullspace: The set of all solutions to $A\vec{x} = \vec{0}$ is called the Nullspace of A.

→ One type of Vector space.

Nullspaces can help characterize solutions to $A\vec{x} = \vec{b}$

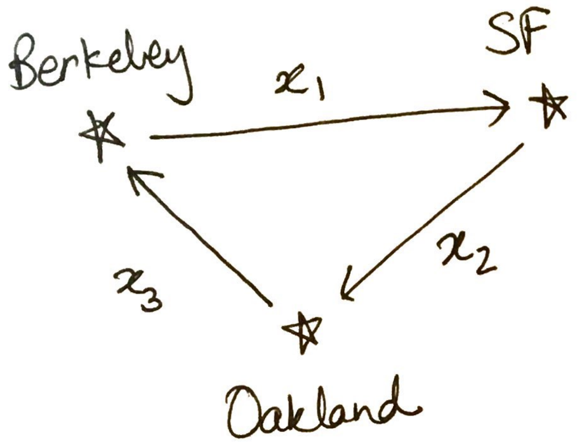
Say $\vec{x_0}$ is a solution to $A\vec{x} = \vec{0}$ (Homogeneous eqⁿ).
$\vec{x_1}$ is a solⁿ to $A\vec{x} = \vec{b}$ (Particular eqⁿ)

$\vec{u} = \vec{x_1} + \alpha \cdot \vec{x_0}$ is also a solⁿ to $A\vec{x} = \vec{b}$

↑ parameter. ~~a~~ any real number.

$$A\vec{u} = A(\vec{x_1} + \alpha \cdot \vec{x_0})$$
$$= \vec{b} + \alpha \cdot A \cdot \vec{x_0} = \vec{b} + \vec{0} = \vec{b}$$

Example: Modeling Traffic flows. (Charge...)



How many measurements do we need to make?

$$x_1 - x_2 = 0$$
$$x_2 - x_3 = 0$$
$$x_3 - x_1 = 0$$

Assume: No cars stop in cities, No accumulation.

$$A \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

leading 1 (pivot)

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

No leading 1 in this column.

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \end{array}$$

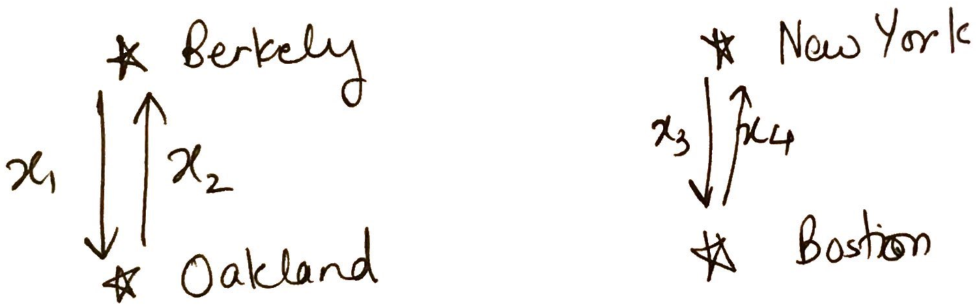
leading 1's

Pivot

$x_3 = t \implies x_1 = t, \; x_2 = t.$

- all vectors $\vec{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$ are solutions.

$$\text{Nullspace}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ ~~is a~~ are solutions.



$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ -x_3 + x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{Row reduction}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Example: $(\mathbb{R}^2, \mathbb{R})$ — Vector space.

Say $(\mathbb{V}, \mathbb{F})$.

Basis: Set of vectors $\{\vec{v_1}, \vec{v_2} \dots \vec{v_n}\}$.

- $\vec{v_1}, \vec{v_2} \dots \vec{v_n}$ are linearly independent
- all $\vec{v} \in \mathbb{V}$ are in the span $\{\vec{v_1}, \vec{v_2} \dots \vec{v_n}\}$.

Minimum spanning set

Dimension of VS

# of elements of the basis

$(\mathbb{R}^2, \mathbb{R})$

Set = $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ Is a basis!

Set = $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$ ✓

$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \rightarrow$ NOT A BASIS.