EECS 16A

• Extra Session: 521 Cory.

Lecture 8

Sept 23, 2019

Last time: Vectorspaces, Basis.

• Nullspace.

• Columnspace = Span = Range.

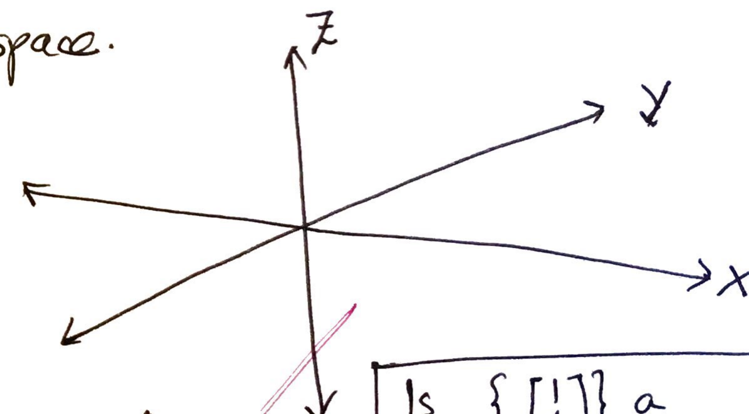
Today:

• Subspace.
• Rank of a matrix.
• Determinants.
• Eigenvalues + Eigenspaces.

• Subspace: $(\mathbb{V}, \mathbb{F})$ is a vector space.

$(\mathbb{W}, \mathbb{F})$ is a subspace if $\mathbb{W}$ is a subset of $\mathbb{V}$

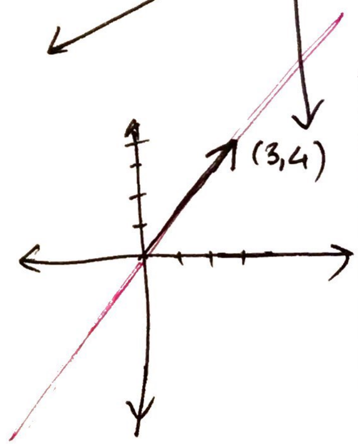
and $(\mathbb{W}, \mathbb{F})$ is also a vector space.



eg. span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^3$

e.g. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

e.g. span $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is this a subspace of $\mathbb{R}^2$



e.g. $\{\vec{0}\}$. a subspace of $\mathbb{R}^2$ $\quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^2$?

- NOT a subspace.

Nullspace (A) : Solutions to $A\vec{x} = \vec{0}$

- Columnspace of (A) = span of the columns of A.

What is the dimension of the columnspace?

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\vec{a_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = \mathbb{R}^2$.

# of elements in Basis = Dimension. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Columnspace has dimension 1.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \; 3 \\ 2 & 4 & 4 \; 7 \end{bmatrix}$ " " " 2

Definition : Rank of a matrix = maximum number of linearly independent columns

$A = \begin{bmatrix} 1 & 3 & 8 & 121 \\ 2 & 4 & 9 & 13 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$

$A \cdot \vec{x} = \vec{b}$

$\uparrow$ 2X1

$A = \begin{pmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 2 & 4 & 2 & 6 \end{pmatrix}$

- Matrix A is invertible $\iff$ A has linearly indep columns.

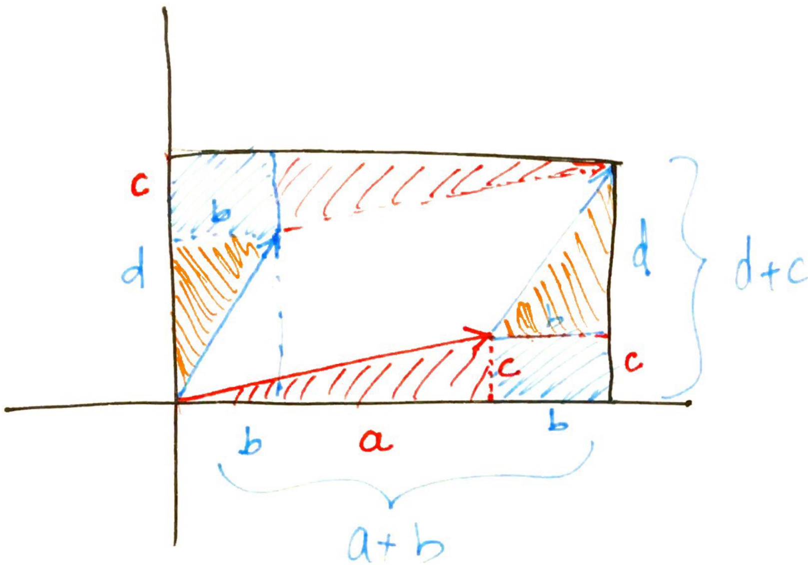
  ↳ matrix is "full-rank"

  $\iff$ $A\vec{x} = \vec{b}$ has a unique solution.

  $\iff$ Nullspace $(A) = \{\vec{0}\}$ is trivial.

  $\iff$ Determinant $(A) \neq 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



Area of big rectangle $= (a+b)(d+c)$

rectangle $= bc + bc$

triangles $= \frac{1}{2} \cdot a \cdot c + \frac{1}{2} \cdot a c.$

triangles $= \frac{1}{2} \cdot bd + \frac{1}{2} \cdot bd.$

Areac of parallelogram $= (a+b)(d+c) - 2bc - ac - bd$

$= ad + ac + bd + bc - 2bc - ac - bd$

$= ad - bc$

$\boxed{\text{Determinant}}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \text{What is it's inverse?}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1/a} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 - c \cdot R_1} \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & d - \frac{b}{a} \cdot c & \frac{-c}{a} & 1 \end{array}\right]$$

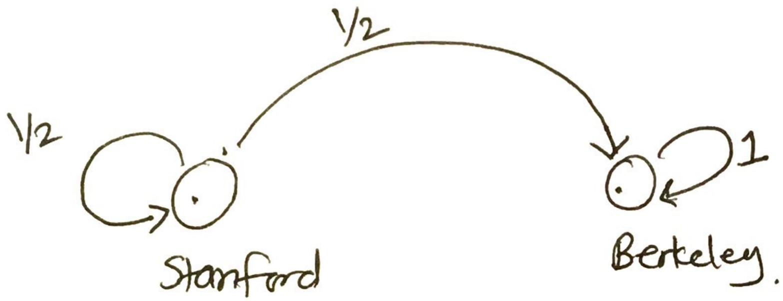
$$= \frac{ad-bc}{a}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \longrightarrow \cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

• Tale of two websites : Page Rank



$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_{stanf} \\ x_{Berk} \end{bmatrix} \qquad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\vec{x}(1) = Q \cdot \vec{x}(0) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{x}(2) = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}, \quad \vec{x}(3) = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} \quad \cdots \quad \vec{x}(t) = \begin{bmatrix} \left(\frac{1}{2}\right)^t \\ 1 - \left(\frac{1}{2}\right)^t \end{bmatrix}$$

$$\vec{x}(\infty) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\quad \vec{x}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ "Steady state"

$\vec{x}_{steady}$

$$\boxed{\vec{x}_{steady} = Q \cdot \vec{x}_{steady}.}$$

$$Q \cdot \vec{x} = 1 \cdot \vec{x}$$

$$Q \cdot \vec{x} - \vec{x} = \vec{0}$$

$$Q \cdot \vec{x} - I \cdot \vec{x} = \vec{0}$$

$$\underbrace{(Q - I)}_{\text{Matrix.}} \cdot \vec{x} = \vec{0}$$

$\vec{x}$ is said to belong to the nullspace of $(Q-I)$.

and it is an eigenvector, belonging to the eigenspace corresponding to eigenvalue 1.