- EECS 16A
  Lecture 9.
  Sept 25, 2018.

- Midterm
- Google: \$739 billion

Last time:

- Determinants.
- Eigenvalue / Eigenspace / Eigenvector.

- Determinant $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$

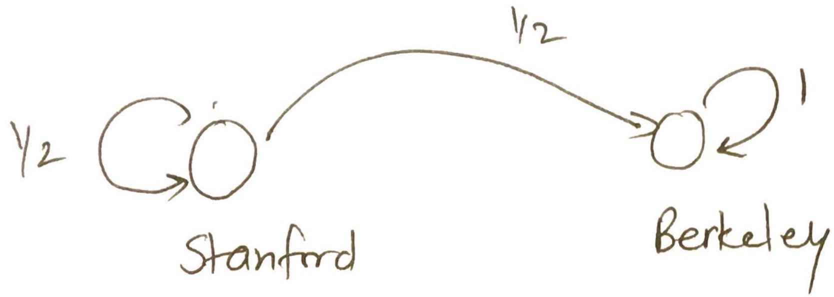
$$ad - bc = 0 \implies ad = bc$$

$$\implies \frac{a}{c} = \frac{b}{d} = \alpha \implies \begin{matrix} \cancel{c = \alpha a} & a = \alpha c \\ \cancel{d = \alpha b}. & b = \alpha d \end{matrix}$$

$$\cancel{\begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}}$$

$$\begin{bmatrix} \alpha c & \alpha d \\ c & d \end{bmatrix}$$

Tale of two websites



$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \vec{x}(t) = \begin{bmatrix} (1/2)^t \\ 1-(\frac{1}{2})^t \end{bmatrix}$

$\vec{x}(t+1) = Q \cdot \vec{x}(t)$. $\quad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

$t \to \infty \quad \vec{x}(t) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

What if we started at $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Q \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_{steady} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a vector such that $Q \cdot \vec{x}_{steady} = \vec{x}_{steady}$.

- General problem:

$$Q \cdot \vec{x} = \vec{x} \qquad (\vec{x} \text{ is an invariant direction}).$$

$$Q\vec{x} - \vec{x} = \vec{0}$$

$$(Q - I)\vec{x} = \vec{0}$$

All $\vec{x} \in \text{Null}(Q-I)$ satisfy this.

Going back: $Q - I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

Compute Null $(Q-I)$

$$\left[\begin{array}{cc|c} -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 + 0 = 0 \\ x_1 = 0 \end{array}$$

free.

$x_2 = t$

$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ is a solution.

$$\text{Null}(Q-I) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

- Eigenspace of the matrix Q, corresponding to eigenvalue 1

More generally: Matrix Q (Square).

$$Q \cdot \vec{x} = \lambda \cdot \vec{x} \qquad \lambda \in \mathbb{R}$$

lambda

then we call λ an eigenvalue of Q.

And $\vec{x}$ is an eigenvector of Q, corresponding to the eigenvalue λ

$\vec{x}$ belongs to the eigenspace corresponding to the eigenvalue λ.

$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

How do we find eigenvalues + eigenvectors?

Want to find: $\lambda, \vec{x}$ such that

$$Q \cdot \vec{x} = \lambda \cdot \vec{x}$$

$$Q \cdot \vec{x} - \lambda \cdot I \cdot \vec{x} = 0$$

UNKNOWN!

$$(Q - \lambda I)\vec{x} = 0$$

When is the nullspace $(Q - \lambda I)$ non-trivial?

↳ i.e. when is it bigger than just $\{\vec{0}\}$.

Det $= 0 \iff$ Nullspace is non-trivial

$\iff$ Matrix is non invertible.

Idea: Use the determinant.

$$Q - \lambda I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1-\lambda \end{bmatrix}$$

$$\text{Det}(Q - \lambda I) = \left(\tfrac{1}{2} - \lambda\right)(1 - \lambda) - \left(\tfrac{1}{2}\right) \cdot 0. = \tfrac{1}{2} - \tfrac{3}{2}\lambda + \lambda^2 = 0.$$

$$\left(\tfrac{1}{2} - \lambda\right)(1 - \lambda) = 0$$

$\lambda_1 = \frac{1}{2}, \lambda_2 = 1$ are solutions.

· $\lambda_2 = \cancel{0}1$ is an eigenvalue.

$$\text{Null}(Q - \lambda_2 I) = \text{Null}\left\{\begin{bmatrix} 1/2 - 1 & 0 \\ 1/2 & 1-1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \longrightarrow \quad \ldots \quad \vec{v}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ is in null}(Q - \lambda_2 I)$$

Therefore, $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is the eigenspace corresponding to $\lambda_2 = 1$.

If eigenvalue $= 1$, the vectors in the eigenspace are in "steady state"

$$Q \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (1)^t = 1$$

$\lambda_1 = \frac{1}{2}$.

$$(Q - \lambda_1 I) = \begin{bmatrix} 1/2 - \lambda_1 & 0 \\ 1/2 & 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Find nullspace:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + x_2 = 0$$

$$x_1 = -x_2$$

Vectors $\in$ Null $(Q - \lambda_1 I)$ = Span$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

$$\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad Q \cdot \vec{w} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \vec{w}$$

$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad Q \cdot \vec{w} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$Q(Q \cdot \vec{w}) = Q(\tfrac{1}{2} \vec{w}) = \tfrac{1}{4} \vec{w}$$

"transient"

$$Q^t \cdot \vec{w} = (\tfrac{1}{2})^t \cdot \vec{w}$$

- $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

① Consider: $(A - \lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$

② $\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8$

③ set $\det = 0$, to find eigenvalues.

$$3 - 4\lambda + \lambda^2 - 8 = 0.$$

$$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda - 5)(\lambda + 1) = 0.$$

Eigenvalues : $\lambda_1 = 5, \ \lambda_2 = -1$

④ Find Null $(A - 5I)$

$$A - 5I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 =$ free

$$x_1 - \frac{x_2}{2} = 0$$

$$x_1 = \frac{x_2}{2}.$$

$$\text{Eigenspace} = \text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

Any vector in this span eigenvector.

- $\vec{v} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$$A \cdot \vec{v} = 5\vec{v}$$
$$A^2 \cdot \vec{v} = 25\,\vec{v}$$
$$A^{100}\,\vec{v} = 5^{100}\,\vec{v}.$$

"blowing up"

---

(5) $\text{Null}(A - (-1)I) = \text{Null}(A + I).$

$$A + I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$\text{Null}(A+I)$ $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + x_2 = 0$

x_2 is free.

Vectors : $\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$ are in the eigenspace.

$\Rightarrow$ Eigenspace corresponding to eigenvalue $(-1) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$$A \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$