EECS 16A

Oct 1, 2019

Lecture 10

- Midterm 10/7
- Review Thursday. evenin

Today:

- More properties of e-values, evectors
- $\vec{x}(t+1) = A\vec{x}(t)$
- Imaging lab 3

Q.1 What if we initialize with an arbitrary $\vec{x}(0)$?

Q2. How do we improve imaging performance?

Thm: A $n \times n$ matrix. $\lambda_1, \lambda_2, \cdots \lambda_n$ be eigenvalues of A.

Eigenvalues distinct $\lambda_1 \neq \lambda_2, \lambda_1 \neq \lambda_3 \cdots \lambda_1 \neq \lambda_n \cdots$

$\lambda_i \neq \lambda_j$ for all i,j $\quad \lambda_2 \neq \lambda_3 \cdots$

$\vec{v_1}, \vec{v_2} \cdots \vec{v_n}$ are the eigenvectors.

If all e-values are distinct, then $\vec{v_1}, \vec{v_2} \cdots \vec{v_n}$ form a basis for $\mathbb{R}^n$.

- Thm: A 2×2. Prove for this case.
  λ_1, λ_2 $\lambda_1 \neq \lambda_2$. e-vectors $\vec{v_1}, \vec{v_2} \rightarrow$ form basis for $\mathbb{R}^2$

Proof:

Know/Given:

$A\vec{v_1} = \lambda_1 \vec{v_1}$ ①

$A\vec{v_2} = \lambda_2 \cdot \vec{v_2}$ ②

$\lambda_1 \neq \lambda_2$

Want to show:

$\vec{v_1}, \vec{v_2}$ form a basis for $\mathbb{R}^2$.

FOCUS $\longrightarrow$ ① $\vec{v_1}, \vec{v_2} \rightarrow$ lin indep..

② $\vec{v_1}, \vec{v_2} \rightarrow$ span $\mathbb{R}^2$..

If possible, assume $\vec{v_1}, \vec{v_2}$ are linearly dependent.

$$\vec{v_1} = \alpha \cdot \vec{v_2} \qquad \alpha \neq 0$$

$$A\vec{v_1} = A \cdot (\alpha \cdot \vec{v_2}) = \alpha \cdot \underbrace{(A \cdot \vec{v_2})}_{\text{from ②}} = \alpha \cdot \lambda_2 \cdot \vec{v_2}.$$

$$A\vec{v_1} = \lambda_1 \cdot \vec{v_1}$$
$$= \lambda_1 \cdot \alpha \cdot \vec{v_2}$$

$$\lambda_1 \alpha \cdot \vec{v_2} = \lambda_2 \cdot \alpha \cdot \vec{v_2}$$

$\Rightarrow$ Contradiction.

Hence, $\vec{v_1}, \vec{v_2}$ must be linearly independent

$\Rightarrow$ They span $\mathbb{R}^2$. $\rightarrow$ They form a basis.

Understand general initial states.

$\vec{x}(t+1) = A \cdot \vec{x}(t)$. Dynamical system.

$\vec{x}(0)$ is in steady state, then state does not change.

$\vec{x} \to$ arbitrary inital state.

Write $\vec{x}$ as a linear combination of the eigenvectors.

$$\vec{x} = \alpha_1 \vec{v_1} + \alpha_2 \cdot \vec{v_2} + \cdots + \alpha_n \vec{v_n}$$

$\vec{v_1}, \vec{v_2} \ldots \vec{v_n}$ are eigenvectors.

λ_i's are distinct.

Why is this possible?

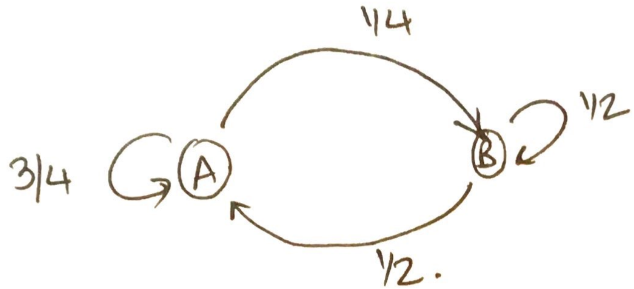
$\{\vec{v_1}, \vec{v_2} \cdots \vec{v_n}\}$ form a basis for $\mathbb{R}^n$.

$$A\vec{x} = A(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n})$$
$$= \alpha_1 A\vec{v_1} + \alpha_2 A\vec{v_2} + \cdots + \alpha_n A\vec{v_n}$$
$$= \alpha_1 \lambda_1 \vec{v_1} + \cdots + \alpha_n \lambda_n \vec{v_n}$$
$$A^2\vec{x} = \alpha_1 \lambda_1^2 \vec{v_1} + \cdots + \alpha_n \lambda_n^2 \vec{v_n}$$
$$A^t\vec{x} = \alpha_1 \lambda_1^t \vec{v_1} + \cdots + \alpha_n \lambda_n^t \vec{v_n}$$

$\lambda_i = 1$

$\lambda_i > 1$

$\lambda_i < 1$



$$Q = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

$\vec{x}(0)$

$$\det(Q - \lambda I) = \left(\tfrac{3}{4} - \lambda\right)\left(\tfrac{1}{2} - \lambda\right) - \frac{1}{8} = \lambda^2 - \frac{5}{4}\lambda + \frac{1}{4}$$

$$= (\lambda - 1)\left(\lambda - \tfrac{1}{4}\right)$$

$\vec{v}_1, \vec{v}_2$ are a basis.

Eigenvalues: $\lambda_1 = 1, \ \lambda_2 = \frac{1}{4}$

$\lambda_1 = 1, \ \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = \frac{1}{4}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\vec{x}(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\vec{x}(0) = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\vec{x}(t) = A^t \cdot \vec{x}(0) = \alpha_1 \cdot \lambda_1^t \, \vec{v}_1 + \alpha_2 \cdot \lambda_2^t \cdot \vec{v}_2$$

$$= \alpha_1 \cdot 1 \cdot \vec{v}_1 + \underbrace{\alpha_2 \left(\tfrac{1}{4}\right)^t \cdot \vec{v}_2}$$

tends to 0 as $t \to \infty$.

As $t \to \infty$

$$A^t \cdot \vec{x}(0) \longrightarrow \alpha_1 \cdot \vec{v}_1 = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Distinct eigenvalues.

$$I\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1, \lambda_2 = 1$ $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Repeated e-val 1. $\vec{v_1}, \vec{v_2}$ form a basis for $\mathbb{R}^2$.

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2$ | $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_1 = 0, \lambda_2 = 0.$

$$A \cdot \vec{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{v_1}$$

$\vec{v_1}$ is an e-vector corresponding to e-value 0.

Eigenvalue $= 0$ $A\vec{v} = 0 \cdot \vec{v} = 0$

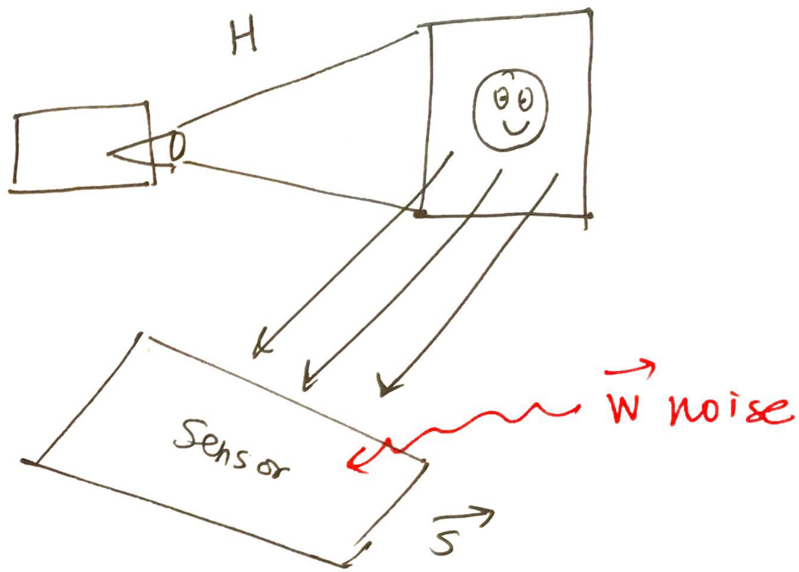
$\Rightarrow$ $\vec{v}$ is in Null(A).

$\Rightarrow$ Columns of A are linearly dep.

A is not invertible.

- Small eigenvalues versus large eigenvalues.

$\vec{i}$ = unknown image.



$$H \cdot \vec{i} = \vec{s}$$

$$\vec{i} = H^{-1} \vec{s}$$

$$\vec{s} = H\vec{i} + \vec{w} \quad \text{(noise)}$$

$$H^{-1}\vec{s} = H^{-1} \cdot H \cdot \vec{i} + H^{-1}\vec{w}$$

$$= \vec{i} + H^{-1} \cdot \vec{w}$$

Want: $H^{-1}\vec{w}$ to be small.

This is connected to the eigenvalues of H^{-1} being small.

$\Rightarrow$ Eigenvalues of H to be large.