Module 3 - Lecture 3

Annoucements

- Optimal Lab.
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① Which satellite is transmitting

→ Inner products

② Distance / Propagation delay.

- Correlation.

  Moving inner product.

③ Distances ⟷ location.

- Trilateration algorithm.

④ Dealing with noise.

- Least squares algorithm
- Projection.

① $\| \vec{x} - \vec{a} \|^2 = d_1^2$

② $\| \vec{x} - \vec{b} \|^2 = d_2^2$

③ $\| \vec{x} - \vec{c} \|^2 = d_3^2$

① $(\vec{x} - \vec{a})^T (\vec{x} - \vec{a}) = d_1^2$

$(\vec{x}^T - \vec{a}^T)(\vec{x} - \vec{a}) = d_1^2$

$\| \vec{x} \|^2 + \| \vec{a} \|^2 - \vec{x}^T \cdot \vec{a} - \vec{a}^T \vec{x} = d_1^2$

④ $\| \vec{x} \|^2 + \| \vec{a} \|^2 - 2 \langle \vec{x}, \vec{a} \rangle = d_1^2$

⑤ $\| \vec{x} \|^2 + \| \vec{b} \|^2 - 2 \langle \vec{x}, \vec{b} \rangle = d_2^2$

⑥ $\| \vec{x} \|^2 + \| \vec{c} \|^2 - 2 \langle \vec{x}, \vec{c} \rangle = d_3^2$

④ - ⑤ $\Rightarrow$ $\| \vec{a} \|^2 - \| \vec{b} \|^2 - 2 \langle \vec{x}, \vec{a} \rangle + 2 \langle \vec{x}, \vec{b} \rangle = d_1^2 - d_2^2$.

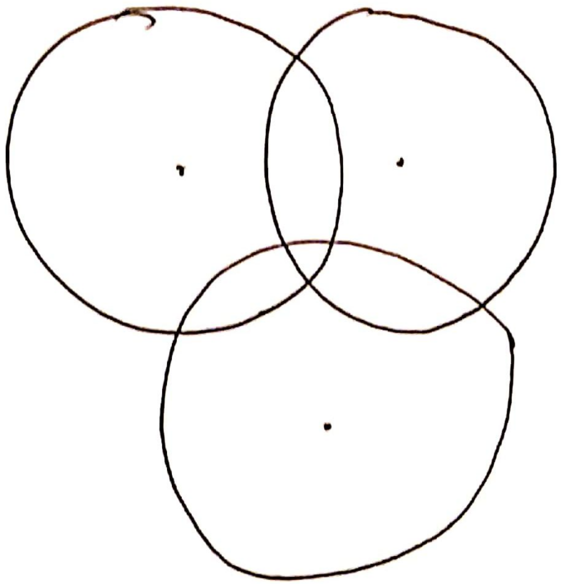
④ - ⑥

$\| \vec{a} \|^2 - \| \vec{b} \|^2 - 2 \vec{a}^T \cdot \vec{x} + 2 \vec{b}^T \cdot \vec{x} = d_1^2 - d_2^2$

Manipulate:

$$\begin{bmatrix} 2(\vec{b}^T - \vec{a}^T) \\ 2(\vec{c}^T - \vec{a}^T) \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} d_1^2 - d_2^2 - \| \vec{a} \|^2 + \| \vec{b} \|^2 \\ d_1^2 - d_3^2 - \| \vec{a} \|^2 + \| \vec{c} \|^2 \end{bmatrix}$$

• What if



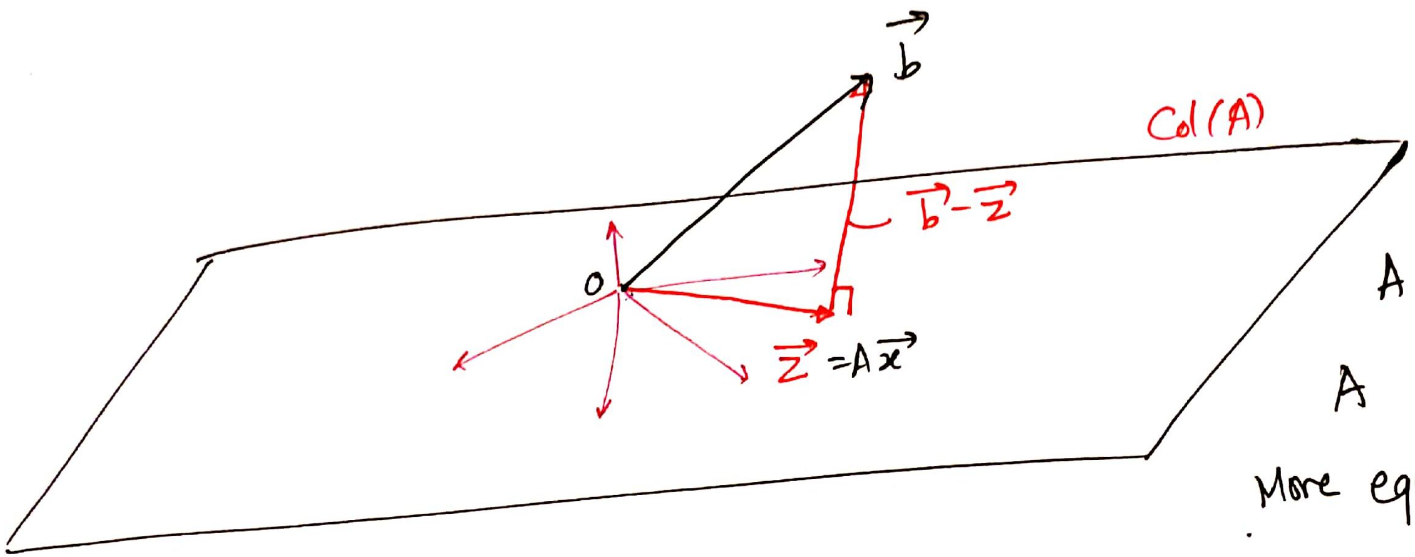
<u>System of equations</u>

$$A\vec{x} = \vec{b}$$

$A\vec{x}$ is approxmitaly $\vec{b}$.

Find $\vec{x}$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible.

$A\vec{x} \in$ Columnspace (A).



Interesting cases:

A is not square.

A : $m \times n$ matrix

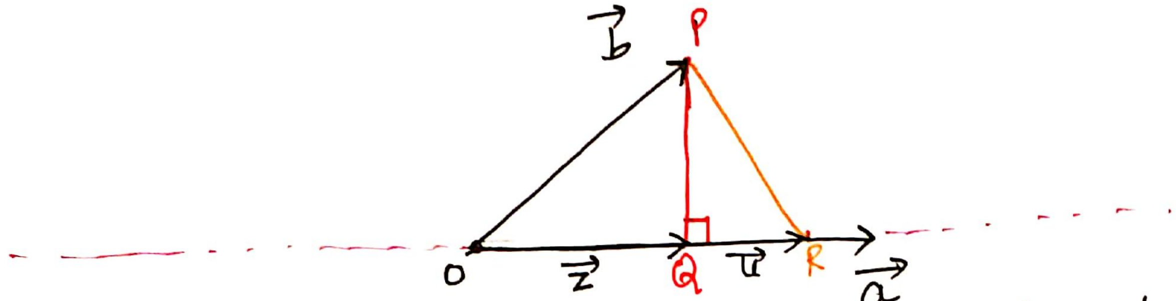
More equations than unknowns.

• Simple case

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{a} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

2x1 matrix

We can move span$\{\vec{a}\}$.



Thm: Shortest distance between a point and a line is given by the perpendicular from the point to the line.

Proof: $\vec{z} = \alpha \cdot \vec{a}$ $\qquad \vec{z} \in \text{span}\{\vec{a}\}$.

scalar

Say R is closer to P than Q is.

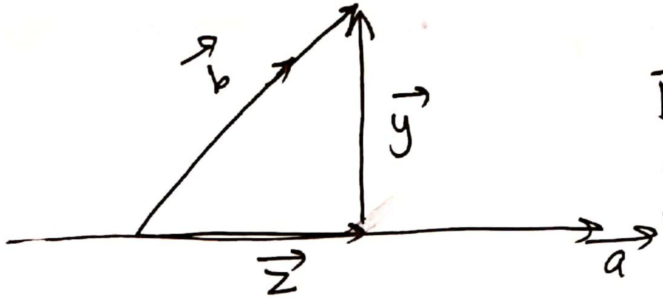
Pythagoras: $PQ^2 + QR^2 = PR^2 \implies \text{len}(PR) > \text{len}(PQ)$

Hypotenuse is the longest side. $\underbrace{\|\vec{b} - \vec{z}\|^2}_{PQ^2} < \underbrace{\|\vec{b} - \vec{u}\|^2}_{PR^2}$

Hence, $\vec{z}$ the closest point.

$\vec{z}$: Called the projection of $\vec{b}$ onto $\vec{a}$

- How do we find α ?



$\vec{b} = \vec{z} + \vec{y}$

$\vec{y} = \vec{b} - \vec{z}$

$\vec{z} = \alpha \cdot \vec{a}$

$$\langle \vec{a}, \vec{y} \rangle = 0$$

$$\langle \vec{a}, \vec{b} - \vec{z} \rangle = 0$$

$$\vec{a}^T (\vec{b} - \vec{z}) = 0$$

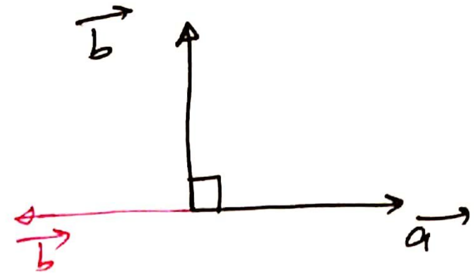
$$\vec{a}^T \cdot \vec{b} - \vec{a}^T \cdot (\alpha \cdot \vec{a}) = 0 .$$

$$\langle \vec{a}, \vec{b} \rangle - \alpha \, \|\vec{a}\|^2 = 0$$

$$\Rightarrow \alpha = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2}$$

Least squares Projection.

Project $\vec{b}$ onto $\vec{a}$.



$\vec{b} = -\vec{a}$

$$\alpha = \frac{\langle \vec{a}, -\vec{a} \rangle}{\|\vec{a}\|^2} = \frac{-\|\vec{a}\|^2}{\|\vec{a}\|^2} = (-1)$$

Moving beyond 2D. Projections generalize.

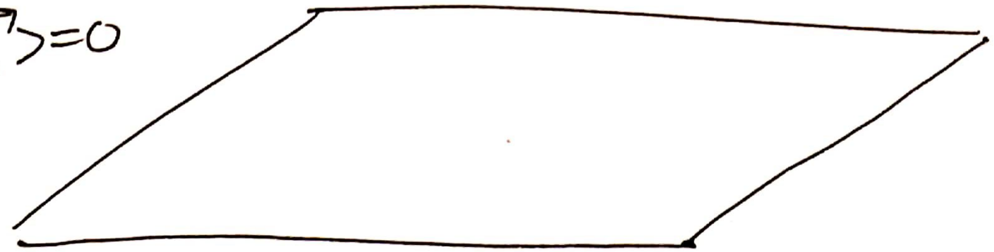
Orthogonality beyond 2D.

Thm: Consider A, and let $\vec{y} \in \mathrm{Col}(A)$.

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1, & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Let $\vec{z}$ be a vector: $\langle \vec{z}, \vec{a}_1 \rangle = 0, \langle \vec{z}, \vec{a}_2 \rangle = 0 \quad \cdots \quad \langle \vec{z}, \vec{a}_n \rangle = 0$

Then, $\langle \vec{z}, \vec{y} \rangle = 0$



Proof:

$\vec{y} \in Col(A)$

$\vec{y}$ is a linear combination of $\vec{a_1}, \dots \vec{a_n}$

$\vec{y} = c_1\vec{a_1} + c_2\vec{a_2} + \dots + c_n \cdot \vec{a_n}$

$\langle \vec{z}, \vec{y} \rangle = \langle \vec{z}, c_1\vec{a_1} + \dots + c_n\vec{a_n} \rangle$

$= \langle \vec{z}, c_1\vec{a_1} \rangle + \langle \vec{z}, c_2\vec{a_2} \rangle + \dots + \langle \vec{z}, c_n\vec{a_n} \rangle$

$= c_1 \langle \vec{z}, \vec{a_1} \rangle + \dots + c_n \langle \vec{z}, \vec{a_n} \rangle$

$= 0$

QED.

$A\vec{x} \underset{\text{approx. eq.}}{\approx} \vec{b}$

A : $m \times n$ matrix
m equations
n unknowns
$m > n$.

Noisy equations

<u>Goal</u>: Find $\vec{x}$ such that $A\vec{x}$ is close to $\vec{b}$.

ie. $\|\vec{e}\|^2 = \|A\vec{x} - \vec{b}\|^2$

$A = \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix}$

<u>Want to find</u>: The orthogonal projection of $\vec{b}$ onto col(A) : Call this $\vec{z}$

$A\vec{x} = \vec{z}$

$\langle (\vec{b} - \vec{z}), \vec{a_1} \rangle = 0$

$\langle b - \vec{z}, \vec{a_2} \rangle = 0$

$\vdots$

$\langle \vec{b} - \vec{z}, \vec{a_n} \rangle = 0.$

$$\vec{a}_1^{\,T}(\vec{b}-\vec{z}) = 0$$

$$\vdots$$

$$\vec{a}_n^{\,T}(\vec{b}-\vec{z}) = 0$$

$$A^T = \begin{bmatrix} - & \vec{a}_1^{\,T} & - \\ & \vdots & \\ - & \vec{a}_n^{\,T} & - \end{bmatrix}$$

$$A^T(\vec{b}-\vec{z}) = 0$$

$$A^T \cdot (\vec{b} - A\vec{x}) = 0$$

$$A^T \cdot \vec{b} - A^T \cdot A\vec{x} = 0$$

$$\underbrace{A^T A}_{n\times n}\vec{x} = A^T\vec{b}$$

$$\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{b}.}$$

General Least-Squares Algorithm.

Projection of $\vec{b}$ onto Col(A)