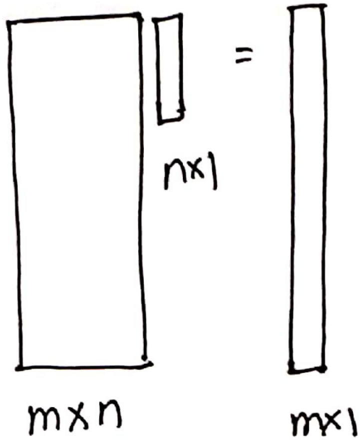
Module 3 - Lecture 4.

Nov 21, 2019.

Least - Squares.

$$A\vec{x} = \vec{b}$$



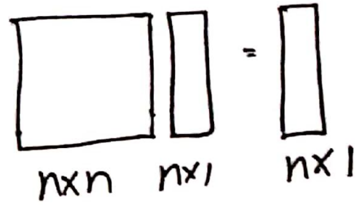
A is not a square mat.

$m > n$

More equations than unknowns

"Over determined system

Square system.



Issue: A is not square

→ A has no inverse.

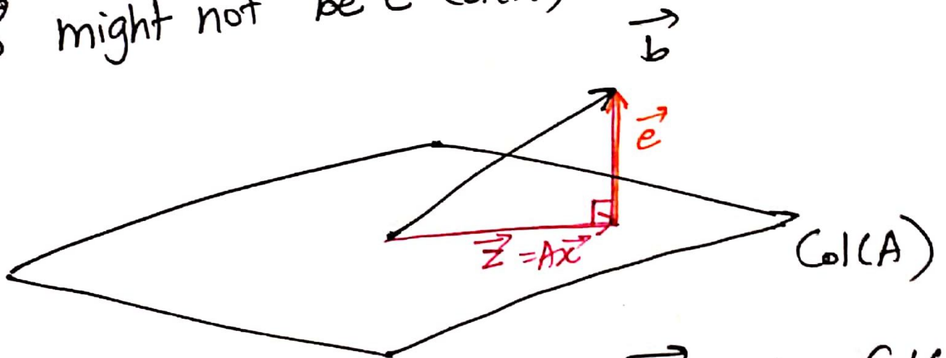
$A\vec{x} \in Col(A)$.

$\vec{b}$ might not be $\in Col(A)$

A^T: $n \times m$

A^TA: $n \times n$

$\hat{\vec{x}}$: best of estimate of $\vec{x}$



The orthogonal projection of $\vec{b}$ onto Col(A) minimizes $\|A\vec{x} - \vec{b}\|^2$ $= \|\vec{e}\|^2$

Solution: "Project" $\vec{b}$ onto col (A).

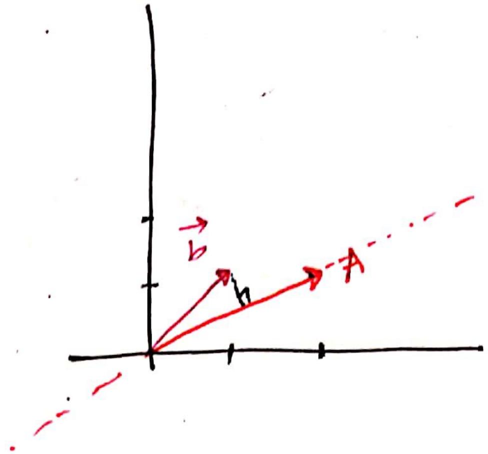
$$\hat{\vec{x}} = (A^TA)^{-1} A^T \cdot \vec{b}$$

$$A\hat{\vec{x}} = A(A^TA)^{-1} A^T \cdot \vec{b}$$

Example:

$$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2×1



$$A \cdot x = \vec{b} \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

① Does GE work?

$$\left[\begin{array}{c|c} 2 & 1 \\ 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{c|c} 1 & 1/2 \\ 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{c|c} 1 & 1/2 \\ 0 & 1/2 \end{array}\right]$$

No solution.

② $\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}.$

$A^T = [2 \quad 1]$ $\qquad A^T A = [2 \quad 1] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$

$(A^T A)^{-1} = 1/5$

$$\hat{\vec{x}} = \frac{1}{5} [2 \quad 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{3}{5}$$

1801 Jan Piazzi 22 observations.

1801 June

Kepler's Law

Planet will have an elliptical orbit.

We know: (x,y) pairs.

$ax^2 + by^2 + cxy + dx + ey = 1$ Eq^n for an ellipse.

Vector of unknowns: $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$

Known: (x_1, y_1) (x_2, y_2) ...

Unknowns: a, b, c, d, e.

$$\underbrace{\begin{bmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 \\ & & \vdots & & \\ x_n^2 & y_n^2 & \cdots & x_n & y_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\vec{b}}$$

$ax_1^2 + by_1^2 + cx_1y_1 + dx_1 + ey_1 = 1$

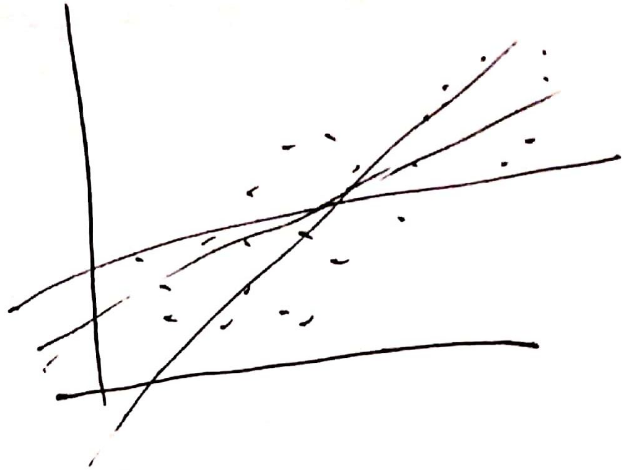
- Linear Regession

$$y = mx + c$$

unknowns.

Line-of-best-fit.

$$\left.\begin{matrix}(x_1, y_1)\\ (x_2, y_2)\\ \vdots \\ (x_n, y_n)\end{matrix}\right\}\ \text{Data}$$



$$y_1 = m\cdot x_1 + c$$

$$y_2 = m\cdot x_2 + c$$

$$\vdots$$

$$\underbrace{\begin{bmatrix}x_1 & 1\\ x_2 & 1\\ \vdots & \vdots\\ x_n & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}m\\ c\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}}_{\vec{b}}$$

- $\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}$

- LS can fail if $A^T A$ is ~~not~~ non-invertible.

$(A^T A)^{-1}$ does not exist $\Longleftrightarrow$ $(A^T A)$ has a non-trivial nullspace.

Thm: Null$(A^T A)$ = Null(A)

Null(A) is non trivial.

$A \cdot \vec{z} = \vec{0}$ → Col of A are linearly dependant.

Proof: (1) If $\vec{v} \in N(A)$, then $\vec{v} \in N(A^T A)$.

(2) If $\vec{w} \in N(ATA)$, then $\vec{w} \in N(A)$

(1) $A \cdot \vec{v} = 0$

Want: $A^T A \cdot \vec{v} = 0$

Mult by $A^T \Rightarrow A^T(A \cdot \vec{v}) = A^T \cdot 0 = 0$.

$\Rightarrow \vec{v} \in N(A^T A)$.

(2) $\vec{w} \in N(A^T A)$

Want: $A\vec{w} = 0$

$A^T A \cdot \vec{w} = 0$

Consider: $A\vec{w} = \vec{q}$

$\langle \vec{q}, \vec{q} \rangle$

$= (A\vec{w})^T (A\vec{w})$ $= \vec{w}^T A^T A \vec{w}$

~~$= \vec{w}^T A^T A \vec{w}$~~

$= \vec{w}^T \cdot 0$

$= 0.$

$\|\vec{q}\| = 0$

Then $\vec{q} = 0$.

$\vec{w} \in N(A)$