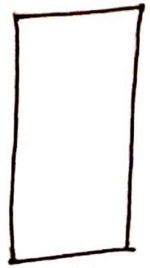
Module 3 - Lecture 5

Nov 26, 2019

Today: Finish Least Squares.

Is A^TA invertible?

$\rightarrow$ Columns of matrix A $\rightarrow$ linearly independent.

A: 

$n \times m$ $\qquad n > m$

- Example A^TA non invertible
- Orthogonal Matching Pursuit.

Last lecture: Thm: $\underbrace{\text{Null}(A^TA)}_{\text{invertibility of } A^TA} = \text{Null}(A)$

Null (A) is non-trivial if columns of A are linearly dependant.

If $\underbrace{A\vec{x}}_{\text{linear comb. of columns.}} = \vec{0}$, $\vec{x} \neq 0$.

Property of transposes. $(AB)^T = B^TA^T$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A: n \times m$ $\qquad AB: n \times k$ $\qquad B^TA^T: k \times n$.

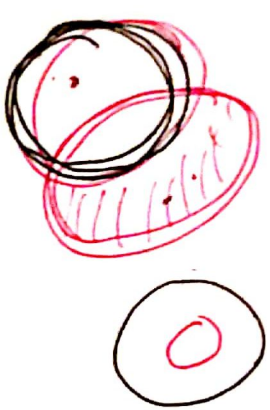
$B: m \times k$ $\qquad A^T: m \times n$

$(AB)^T: k \times n$ $\qquad B^T: k \times m$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Proof: (1) If $\vec{v} \in Null(A)$, then $\vec{v} \in Null(A^TA)$

(2) If $\vec{w} \in Null(A^TA)$, then $\vec{w} \in Null(A)$



(1) $\vec{v} \in Null(A)$

$A \cdot \vec{v} = 0$

Want: $\vec{v} \in N(A^TA)$

$A^TA\,\vec{v} = \vec{0}$

Mult. by A^T.

$A^TA \cdot \vec{v} = A^T \cdot 0 = 0$.

$\Rightarrow \vec{v} \in Null(A^TA)$

(2) $\vec{w} \in N(A^TA)$

$\boxed{A^TA\vec{w} = 0.}$

Want: $\vec{w} \in N(A)$.

$A\vec{w} = 0$

~~eg~~ $\langle x, y \rangle = x^Ty$

Consider:

$$\|A\vec{w}\|^2 = \langle A\vec{w}, A\vec{w} \rangle$$
$$= (A\vec{w})^T A\vec{w} = (\vec{w}^TA^T)A\vec{w} = \vec{w}^T(A^TA\vec{w})$$
$$= \vec{w}^T\underbrace{A^TA\vec{w}}$$
$$= 0 \checkmark$$

The only way this is possible is if $A\vec{w} = \vec{0}$ = $\vec{w} \in N(A)$.

• If $\|\vec{x}\|^2 = \underbrace{x_1^2}_{\geq 0} + \underbrace{x_2^2}_{\geq 0} + \cdots + \underbrace{x_n^2}_{\geq 0} = 0$.

Thm: If $\|\vec{x}\|^2 = 0$, then $\vec{x} = \vec{0}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

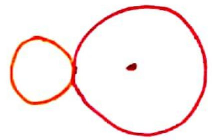
$x_1 = 0, \quad x_2 = 0, \quad \cdots \quad x_n = 0.$

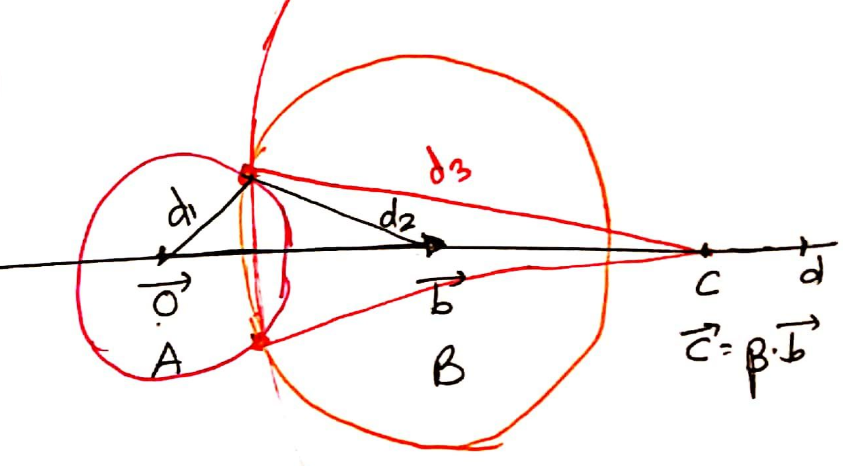
$x_1 = (-1) \quad , x_1^2 = +1.$

• Columns of A $\longleftrightarrow$ Invertibility of A^TA.

$\vec{a} = \vec{0}$, $\vec{b} = \vec{b}$, $\vec{c} = \beta \cdot \vec{b} = 2 \cdot \vec{b}$ $\beta = 2$

$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$





$$\underbrace{2\begin{bmatrix} \vec{b}^T - \vec{a}^T \\ \vec{c}^T - \vec{a}^T \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_1^2 - d_2^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \end{bmatrix}}_{\text{KNOWN.}}$$

$$\begin{bmatrix} \vec{b}^T \\ \vec{c}^T \end{bmatrix} = \begin{bmatrix} \vec{b}^T \\ 2\vec{b}^T \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 & b_2 \\ 2b_1 & 2b_2 \end{bmatrix}}$$

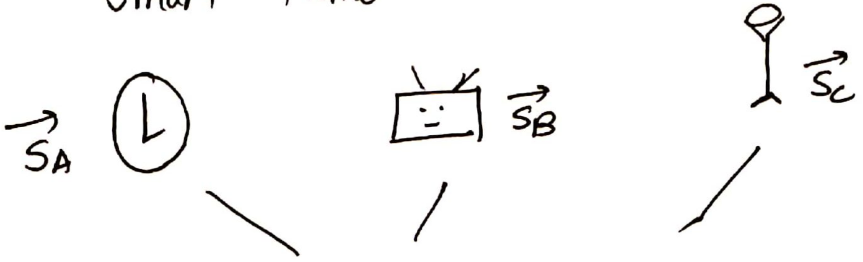
Columns are linearly dep.

~~Is A~~ Are columns of A linearly indep? $\longrightarrow$ NO.

$\Rightarrow$ A^TA invertible? $\longrightarrow$ NO.

Orthogonal Matching Pursuit

"Smart Home"



How do you figure out which devices are transmitting?

10,000s of devices.

Gold codes: $\vec{S_A}$, $\vec{S_B}$. ± 1 Gold codes

1023 length gold codes.

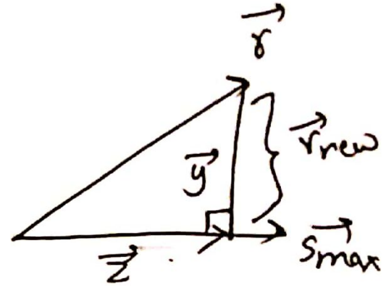
$\langle \vec{S_A}, \vec{S_A} \rangle =$ large
$= N$ (length of vector).

$\langle \vec{S_A}, \vec{S_B} \rangle =$ small, almost orthogonal.

$$\vec{r} = \alpha_1 \vec{S_1} + \alpha_2 \vec{S_2} + \cdots + \alpha_{10,000} \vec{S_{10,000}}$$

$$
\begin{array}{c c}
1023 & \underbrace{\begin{bmatrix} | & | & & | \\ \vec{s}_1 & \vec{s}_2 & \cdots & \vec{s}_{10,000} \\ | & | & & | \end{bmatrix}}_{10,000}
\end{array}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{10,000} \end{bmatrix}
=
\underbrace{\begin{bmatrix} \\ \vec{r} \\ \\ \end{bmatrix}}_{1023}
$$

$$\underline{\underline{\vec{r}}} = \vec{s}_A + 2\vec{s}_B$$



$\vec{z} + \vec{y} = \vec{r}$ $\quad \vec{r} - \vec{z} = \vec{y}$

$\vec{z} = \mathrm{proj}_{\vec{s}_{max}}(\vec{r})$

How to solve this: <u>GREEDY</u>.

Consider:

$\langle \vec{r}, \vec{s}_A \rangle$

$\langle \vec{r}, \vec{s}_B \rangle$

$\vdots$

$\langle \vec{r}, \vec{s}_C \rangle$

$\vdots$

$\langle \vec{r}, \vec{s}_Z \rangle$

$\vdots$

$\langle \vec{r}, \vec{s}_{10,000} \rangle$

- Consider all the inner products.
- Find the maximum inner product: $\vec{s}_{max}$

<u>Guess:</u> $\vec{s}_{max}$ is transmitting.

Project: $\vec{r}$ onto $\vec{s}_{max}$

$$\mathrm{Proj}_{\vec{s}_{max}}(\vec{r})$$

$$\vec{r}_{new} = \vec{r} - \mathrm{proj}_{\vec{s}_{max}}(\vec{r}).$$