# EECS 16A   Designing Information Devices and Systems I
## Fall 2021   Discussion 3A

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.
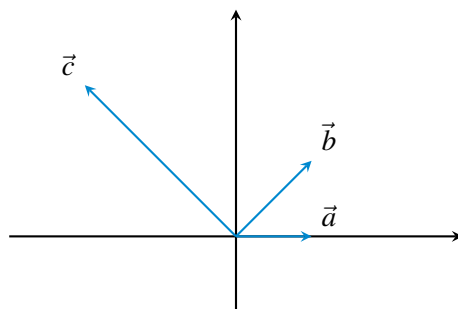
(a) First, consider the case where $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Draw these vectors on a sheet of paper.

Answer:
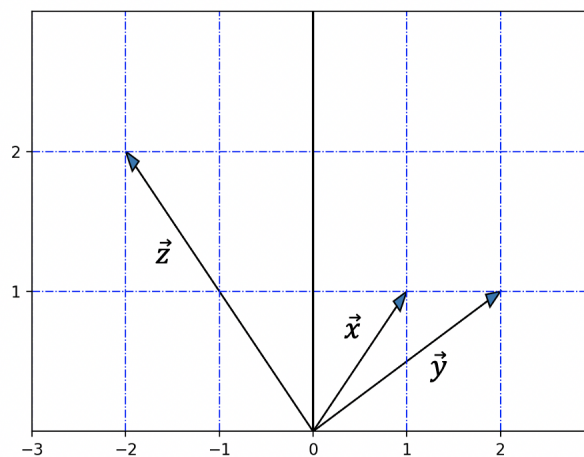
(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

Answer:

$$\alpha\vec{x} + \beta\vec{y} = \vec{z}$$

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{cases} \alpha + \beta \cdot 2 = -2 \\ \alpha + \beta = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(c) Solve for α, β.

Answer: We start by writing the system in the augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array} \right]$$

Then we solve the system using Gaussian Elimination. First, we subtract the second row by the first row:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array} \right]$$

Next, we multiply the second row by -1 to solve for β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array} \right]$$

We get $\beta = -4$. Then, we take the first row and subtract it by the second row*2.

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array} \right]$$

So the solution is $\alpha = 6$ and $\beta = -4$.

2. **Span basics**

(a) What is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We realize that any vector whose last component is 0 can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: Yes. We realize from inspection that

$$\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(c) What is a possible choice for $\vec{v}$ that would make span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some x such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).

3. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is **linearly dependent** if there exists constants $c_1, c_2, \ldots c_n$ such that $\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $A\vec{x} = \vec{0}$. Prove that the columns of A are linearly dependent.

Answer:

Begin by defining column vectors $\vec{a}_1 \ldots \vec{a}_n$.

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix}$$

Thus, we can represent the multiplication $A\vec{x}$ as

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \sum x_i \vec{a}_i = \vec{0}$$

Note that the equation above is the definition of linear dependence. That is, there exist coefficients, at least one which is non-zero, such that the sum of the vectors weighted by the coefficients is zero. These coefficients are the elements of the non-zero vector $\vec{x}$.

(b) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Let us consider the difference of the two equations:

$$\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

Once again, we've reached the definition of linear dependence since $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$. We can apply the results from part (a), setting $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Answer: The key insight is to use the linearity of Matrix-vector multiplication.

By assumption, let $\vec{x}_1 \in \mathbb{R}^n$ be a solution to $\mathbf{A}\vec{x} = \vec{b}$. Then, for any $c \in \mathbb{R}$

$$\mathbf{A}(\vec{x}_1 + c\vec{y}) = \mathbf{A}\vec{x}_1 + \mathbf{A}(c\vec{y}) = \mathbf{A}\vec{x}_1 + c\mathbf{A}\vec{y} = \mathbf{A}\vec{x}_1 + \vec{0} = \mathbf{A}\vec{x}_1 = \vec{b}$$

where the first two equalities follow by linearity and the last two equalities follow from the assumptions that $\mathbf{A}\vec{y} = \vec{0}$ and that $\vec{x}_1$ is a solution to the system.

Hence, $\mathbf{A}(\vec{x}_1 + c\vec{y}) = \vec{b}$, implying that $(\vec{x}_1 + c\vec{y})$ is also a solution to $\mathbf{A}\vec{x} = \vec{b}$ for **any** constant c. Therefore, there are infinitely many solutions.