EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 4B

Recall from lecture the way to compute a determinant of any 2×2 matrix is by using the following formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$

1. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Answer:

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

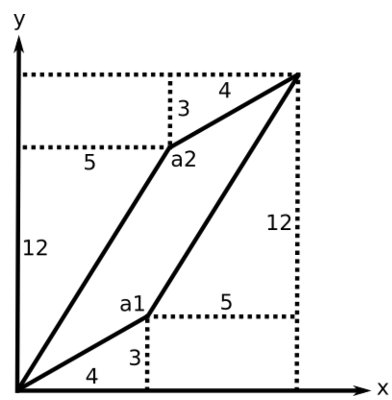
$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 \cdot 0 = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 0 = 6$$

(c) We know that the determinant of a matrix represents the multi-dimensional volume formed by the column vectors. Explain intuitively why the determinant of a matrix with linearly dependent column vectors is always 0.



Answer: Consider an example in $\mathbb{R}^2$. If the vectors are linearly independent, we can form some parallelogram and calculate some nonzero area, and we get a nonzero determinant as expected. If the vectors are linearly dependent though, the "parallelogram" we form ends up only having 1 dimension. The other dimension was compressed to 0, and so we have 0 area, corresponding to the 0 determinant. This idea generalizes to N dimensions. If we have fewer than N linearly independent vectors, then the multi-dimensional volume will have at least 1 dimension compressed to 0, giving us 0 volumen and 0 determinant.

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix **M** and the associated eigenvectors. State if the inverse of **M** exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} -\lambda(-3-\lambda)+2 &= 0 \\ \lambda^2+3\lambda+2 &= 0 \\ (\lambda+2)(\lambda+1) &= 0 \\ \lambda &= -1, -2 \end{aligned}$$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -1$ is $\text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -2$ is $\text{span}\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$.

Note that we have no zero eigenvalues, the columns of A are linearly independent, and the determinant of A is non-zero (evaluate our polynomial in λ at $\lambda = 0$). Any of these are equivalent conditions for saying that a square matrix is invertible.

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} (-2-\lambda)(8-\lambda)+16 &= 0 \\ \lambda^2 - 6\lambda &= 0 \\ \lambda(\lambda-6) &= 0 \\ \lambda &= 0, 6 \end{aligned}$$

$\lambda = 0$:

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -4 & 8 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = 0$ is $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

$\lambda = 6$:

$$\left[\begin{array}{cc|c} -2-6 & 4 & 0 \\ 4 & 8-6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -8 & 4 & 0 \\ -4 & 2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = 6$ is $\text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$.

Matrix $\mathbf{M}$ has linearly dependent columns, therefore the inverse $\mathbf{M}^{-1}$ does not exist. Note also that $\mathbf{M}$ has an eigenvalue of 0 so that $N(\mathbf{M})$ contains more than just $\vec{0}$. For this reason also $\mathbf{M}$ is not invertible.

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} \lambda^2 &= 0 \\ \lambda &= 0(\times 2) \end{aligned}$$

$\lambda = 0$:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ cannot be further reduced by G.E.}$$

$$x_2 = 0, x_1 = t \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

The eigenspace for $\lambda = 0$ is span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Note even though $\lambda = 0$ is a eigenvalue with multiplicity 2 (occurs as a root twice for the characteristic polynomial), the dimension of its eigenspace is only 1. This shows that the number of linearly independent eigenvectors for a given eigenvalue is not necessarily equal to the multiplicity, i.e. the number of times that eigenvalue occurs in the characteristic polynomial.

Matrix $\mathbf{M}$ has a zero column (linearly dependent columns), therefore the inverse $\mathbf{M}^{-1}$ does not exist.

(d) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Answer: Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right) = 0$$

$$\lambda^2 + 1 = 0$$

From the above equation, we know that the eigenvalues are $\lambda = i$ and $\lambda = -i$.

For the eigenvalue $\lambda = i$:

$$(\mathbf{M} - i\mathbf{I})\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{x} = \vec{0}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{x} = \vec{0}$$

We can also perform Gaussian elimination on matrices with imaginary or complex numbers:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{array}{c} x_1 - ix_2 = 0 \\ x_2 = t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} t$$

So the eigenspace is span$\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = -i$:

$$(\mathbf{M} + i\mathbf{I})\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \right) \vec{x} = \vec{0}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{array}{c} x_1 + ix_2 = 0 \\ x_2 = t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} t$$

The second eigenspace is span$\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

(e) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$\begin{aligned} (\mathbf{M} - 1\mathbf{I})\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} &= \vec{0} \end{aligned}$$

From the second equation in the system, $x_2 = 0$, with any solution having the form $\begin{bmatrix} 1 \\ 0 \end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is thus $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$\begin{aligned} (\mathbf{M} - 9\mathbf{I})\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{x} &= \vec{0} \\ \begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} &= \vec{0} \end{aligned}$$

From the first equation in the system, $x_1 = 0$, so any solution must take the form $\begin{bmatrix} 0 \\ 1 \end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

The matrix is invertible.

3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer: Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1 \cdot \vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer: Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

Answer: In a conceptual sense, there are three cases:

Rotation by $0°$: (more accurately, any integer multiple of $360°$), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

Rotation by $180°$: (more accurately, any angle of $180° + n \cdot 360°$ for integer n), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of -1 because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

Any other rotation: there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 1 for a visualization.

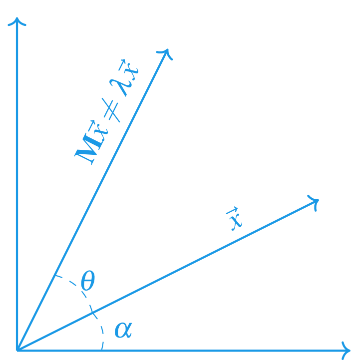


Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of $360°$ (identity matrix) or the rotation angle is $\theta = 180° + n \cdot 360°$ for any integer n ($-\mathbf{I}$).

(d) **(PRACTICE)** Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Answer: Using our known determinant formula for 2x2 matrices $\det(A) = ad - bc$ we can compute the characteristic polynomial

$$\det(\mathbf{R} - \lambda \mathbf{I}) \;=\; \det \begin{bmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix} \;=\; \cos(\theta)^2 + \sin(\theta)^2 - 2\cos(\theta)\lambda + \lambda^2 \;\equiv\; 0$$

From here we can first simplify $1 = \cos(\theta)^2 + \sin(\theta)^2$ and then use the quadratic formula to attain the two possible λ values.

$$\lambda = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1} = \cos(\theta) \pm i\sqrt{1 - \cos(\theta)^2} = \cos(\theta) \pm i\sqrt{\sin(\theta)^2}$$

In exponential phase notation we can write the two eigenvalues more concisely: $\lambda = e^{\pm i\theta}$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Answer: Yes, both $+1$ and -1. Mechanically, we could go through the methods we have learned for attaining a characteristic polynomial from $\det(T - \lambda I) = (1-\lambda)(-1-\lambda) - (0)(0)$ and recalling our eigenvalues are the roots of this polynomial (the values where this polynomial is zero). This works because matrix $T - \lambda I$ only has a nonempty null space when its determinant is zero!

$$\det(T - \lambda I) = \lambda^2 - 1 \equiv 0 \quad \rightarrow \quad \lambda = \pm 1$$

Conceptually, we can reason that a vector along the x-axis will be unaffected by $\mathbf{T}$ (in this case $\lambda = +1$), where as a vector along the y-axis gets perfectly flipped by $\mathbf{T}$ (in this case $\lambda = -1$)

NOTE: A 2×2 reflection matrix always has $\lambda = \pm 1$, REGARDLESS of the axis of reflection. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal (perpendicular) to the reflection axis will just “flip it/negate it” (eigenvalue -1). In other words, the set of vectors that lie along the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the set of vectors orthogonal to the axis of reflection is the eigenspace associated with the eigenvalue -1.

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Answer: $N(A)$ is not just $\vec{0}$ as we have some $\vec{v} \neq \vec{0}$ satisfying $A\vec{v} = \lambda\vec{v}$. Another way we can state this is that $\dim(N(A)) > 0$.
Thus we can imagine if $\mathbf{M}\vec{x} = \vec{b}$ has a solution then $\mathbf{M}(\vec{x} + \vec{v}) = \vec{b}$ also solves the system, hence there are infinite solutions. Yet we also know that a nonzero null space means $\mathbf{M}$ has linearly dependent columns, so the vector $\vec{b}$ could lie outside of this span in which case there is no solution.

In summary, there are either infinite or no solutions to the system of equations $\mathbf{M}\vec{x} = \vec{b}$

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Note that the matrix has linearly dependent columns. Therefore, according to part (f), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

