EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 5B

1. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M}-\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

This means that

$$span\{\vec{v}_1\} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

ii. $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{M}-2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

This means that

$$span\{\vec{v}_2\} = span\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$$

iii. $\lambda = \frac{1}{2}$:

$$\left[\begin{array}{c|c} \mathbf{M} - \frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

This means that

$$span\{\vec{v}_3\} = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned} \mathbf{M}^n \vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3) \\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3 \\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n \gamma\vec{v}_3 \end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

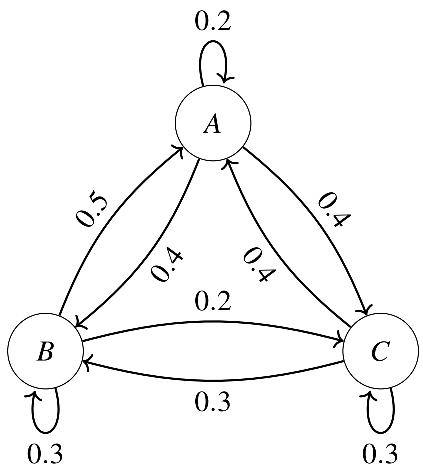

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$\begin{aligned} T\vec{x} &= 1\vec{x} \\ &= \lambda_1\vec{x} \\ \Rightarrow \vec{x} &\in N(\mathbf{T} - 1 \cdot \mathbf{I}) \\ \vec{x} &\in N\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ \vec{x} &\in N\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}\right). \end{aligned}$$

In order to row reduce $\mathbf{T} - 1 \cdot \mathbf{I}$ we use Gaussian elimination. We also convert to fractions:

$$\begin{bmatrix} -\frac{4}{5} & \frac{1}{2} & \frac{2}{5} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \xRightarrow{R_1 \leftarrow -5/4 R_1} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \xRightarrow[R_3 \leftarrow R_3 - 2/5 R_1]{R_2 \leftarrow R_2 - 2/5 R_1} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & -\frac{9}{20} & \frac{1}{2} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix}$$

$$\xRightarrow{R_2 \leftarrow -20/9 R_2} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix} \xRightarrow{R_3 \leftarrow R_3 - 9/20 R_2} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \leftarrow R_1 + 5/8 R_2} \begin{bmatrix} 1 & 0 & -\frac{43}{36} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Thus we can write the form any steady state vector should take using the first two equations represented by the row reduced matrix:

$$\begin{aligned} x_1 - \tfrac{43}{36}x_3 &= 0 \\ x_2 - \tfrac{10}{9}x_3 &= 0 \\ x_3 &= \alpha \in \mathbb{R} \end{aligned} \implies \vec{x} = \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} \alpha$$

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

Answer: The magnitude of both eigenvalues is less than 1, so in steady state, the components associated with those eigenvectors $\vec{v}_2$ and $\vec{v}_3$ will trend toward $\vec{0}$. Additionally, since $\lambda_2 < 0$, its associated eigenvector will oscillate / flip signs back and forth.

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Answer:

From the previous sub-parts we know the steady-state solution should have the form (rescaled for convenience) $\vec{x}_{ss} = \alpha \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}$ for any α.

But after inspecting the transition matrix we recognize that the columns each sum to one, thus we have a conservative system, meaning that the total volume across all three reservoirs $(A_0 + B_0 + C_0)$ must remain constant at all iterations. This gives us a sufficient condition to identify α.

So far the sum, with $\alpha = 1$ of $\vec{x}_{ss}$ is $43 + 40 + 36 = 119$ (kiloliters), while the initial state starts with $A_0 + B_0 + C_0 = 129 + 109 + 0 = 238$ kiloliters. By inspection we see that $\alpha = 2$ is the proper rescaling of the steady-state eigenvector to satisfy this condition. Thus

$$\vec{x}_{ss} = \begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}. \qquad \square$$