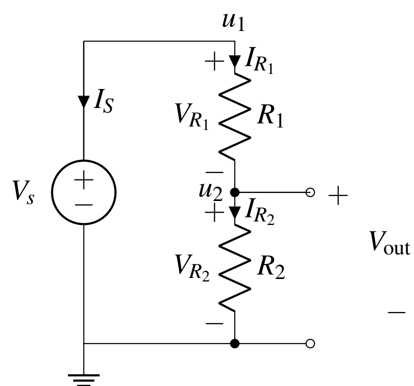
EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 6A

1. Voltage Divider

For the circuit below, your goal will be to find the voltage V_{out} in terms of the resistances R_1, R_2, and V_s, using NVA (Node Voltage Analysis) and Gaussian elimination. The labeling steps (steps 1-4) have already been done for you.



Here is a reminder of the labeling steps followed to get the circuit diagram above:

- **Step 1:** Select a reference (ground) node. Any node can be chosen for this purpose. We will measure all of the voltages in the rest of the circuit relative to this point.
- **Step 2:** Label all nodes with voltage set by voltage sources.
- **Step 3:** Label remaining nodes.
- **Step 4:** Label element voltages and currents, following **Passive Sign Convention**.

Our goal is to ***find*** V_{out}. In order to do this, we can use NVA to find equations describing our circuit, write our equations in the form $\mathbf{A}\vec{x} = \vec{b}$, and use Guassian elemination to solve for $\vec{x}$. The following steps will walk you through this process:

Step 5: Write out $\mathbf{A}\vec{x} = \vec{b}$, leaving the entries for $\mathbf{A}$ and $\vec{b}$ blank. Next, fill in the enteries for $\vec{x}$. Recall that $\vec{x}$ is a vector of your unknown currents and voltages.

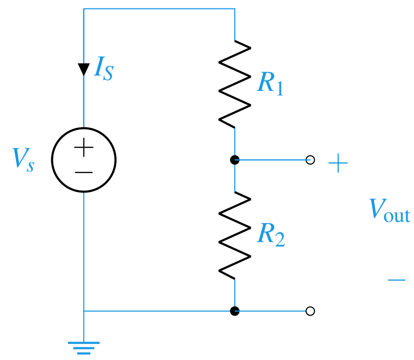
Step 6: Write KCL equations for all nodes with unknown voltages. Using these equations, fill in as many linearly indepedent rows in $\mathbf{A}$ and $\vec{b}$ as possible.

Step 7: Write down the IV relationships (Ohm's Law) of each of the non-wire elements. Use these equations to fill in the remaining rows in $\mathbf{A}$ and $\vec{b}$. (Hint: how many equations do you need to write?)

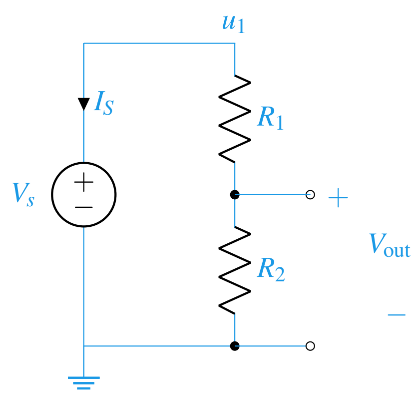
Step 8: Use Gaussian elimination or substition to solve for $u_2 = V_{out}$.

Answer:

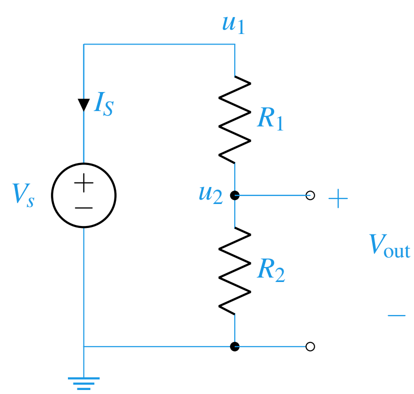
Step 1: Select a ground node,



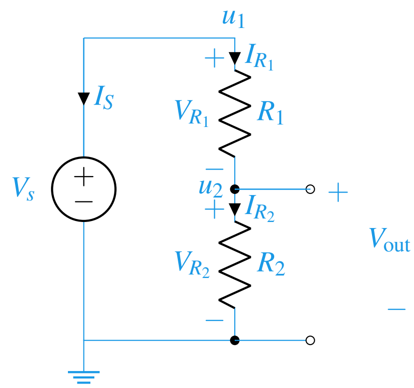
Step 2: Label all nodes with voltage set by voltage sources (denoted below as u_1),



Step 3: Label remaining nodes (denoted below as u_2),



Step 4: Label element voltages and currents following passive sign convention,



Step 5: Write out $\mathbf{A}\vec{x} = \vec{b}$, leaving the entries for $\mathbf{A}$ and $\vec{b}$ blank. Fill in the entries for $\vec{x}$:

$$\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} I_S \\ I_{R_1} \\ I_{R_2} \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Step 6: Write KCL equations for all nodes with unknown voltages. Using these equations, fill in as many linearly indepedent rows in $\mathbf{A}$ and $\vec{b}$ as possible.

$$I_{R_1} = I_{R_2} \Rightarrow I_{R_1} - I_{R_2} = 0$$
$$I_S + I_{R_1} = 0$$
$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} I_S \\ I_{R_1} \\ I_{R_2} \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ? \\ ? \\ ? \end{bmatrix}$$

Step 7: Write down the IV relationships (Ohm's Law) of each of the non-wire elements. Use these equations to fill in the remaining rows in $\mathbf{A}$ and $\vec{b}$.

$$V_s = u_1$$
$$I_{R_1} = \frac{V_{R_1}}{R_1} = \frac{u_1 - u_2}{R_1} \Rightarrow I_{R_1}R_1 - u_1 + u_2 = 0$$
$$I_{R_2} = \frac{V_{R_2}}{R_2} = \frac{u_2 - 0}{R_2} \Rightarrow I_{R_2}R_2 - u_2 = 0$$
$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & R_1 & 0 & -1 & 1 \\ 0 & 0 & R_2 & 0 & -1 \end{bmatrix} \begin{bmatrix} I_S \\ I_{R_1} \\ I_{R_2} \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

Step 8: Use Gaussian elimination or substitution to solve for $u_2 = V_{out}$.

Substitution:

$$\begin{aligned}
&I_{R_2} = I_{R_1} \\
&\Rightarrow \frac{V_s - u_2}{R_1} = \frac{u_2 - 0}{R_2} \\
&\Rightarrow (V_s - u_2)R_2 = u_2 R_1 \\
&\Rightarrow u_2 = \frac{R_2}{R_1 + R_2} V_s
\end{aligned}$$

Gaussian Elimination:

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & R_1 & 0 & -1 & 1 & 0 \\ 0 & 0 & R_2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & R_1 & 0 & -1 & 1 & 0 \\ 0 & 0 & R_2 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & 0 & R_1 & -1 & 1 & 0 \\ 0 & 0 & R_2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & R_1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & 0 & R_2 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{R_1} & \frac{1}{R_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & 0 & 1 & 0 & -\frac{1}{R_2} & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & R_1 & -R_1 & 0 \\ 0 & 1 & 0 & -R_1 & R_1 & 0 \\ 0 & 0 & 1 & -\frac{1}{R_1} & \frac{1}{R_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & 0 & 0 & \frac{1}{R_1} & -\frac{1}{R_2} - \frac{1}{R_1} & 0 \end{array}\right]$$

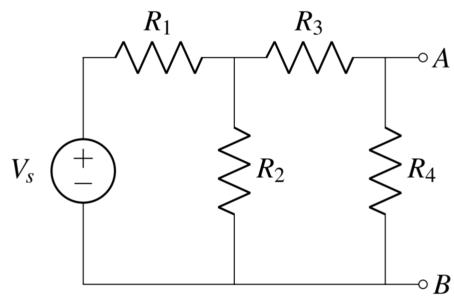
$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & R_1 & -R_1 & 0 \\ 0 & 1 & 0 & -R_1 & R_1 & 0 \\ 0 & 0 & 1 & -\frac{1}{R_1} & \frac{1}{R_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & V_s \\ 0 & 0 & 0 & 0 & -\frac{1}{R_2} - \frac{1}{R_1} & -V_s\frac{1}{R_1} \end{array}\right]$$

At this point, we can stop our Guassian elimination because we only need to solve for u_2:

$$\begin{aligned}
u_2 \frac{1}{R_2} + \frac{1}{R_1} &= V_s \frac{1}{R_1} \\
u_2 &= V_s \frac{1}{R_1} \frac{1}{\frac{1}{R_2} + \frac{1}{R_1}} \\
u_2 &= V_s \frac{1}{R_1} \frac{1}{\frac{R_1 + R_2}{R_1 R_2}} \\
u_2 &= V_s \frac{R_2}{R_1 + R_2}
\end{aligned}$$

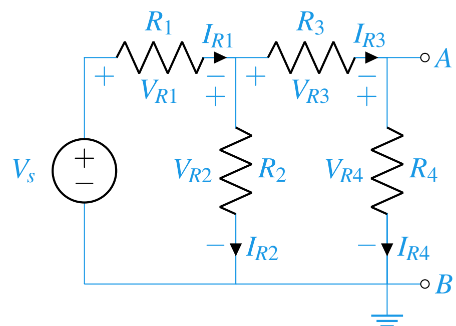
2. KVL and KCL

For the circuit shown below, $V_s = 5\,\text{V}$, $R_1 = R_2 = 4\,\text{k}\Omega$, and $R_3 = R_4 = 2\,\text{k}\Omega$.



(a) For the circuit above, write KVL equations for each loop and KCL equations for each node.

Answer:



KVL through the three loops:

$$V_s - V_{R_1} - V_{R_2} = 0$$

$$V_s - V_{R_1} - V_{R_3} - V_{R_4} = 0$$

$$V_{R_2} - V_{R_3} - V_{R_4} = 0$$

KCL at each of the nodes:

$$I_{V_s} + I_{R_1} = 0$$

$$-I_{R_1} + I_{R_2} + I_{R_3} = 0$$

$$-I_{R_3} + I_{R_4} = 0$$

$$-I_{R_4} - I_{R_2} - I_{V_s} = 0$$

The above equations combined with Ohm's law for each resistor would let us find the current through and voltage across each resistor. The system would in general be overdetermined.

(b) Solve for the voltage between A and B using the equations from part (a).

Answer:

From the IV relations of the resistors, the KVL equations in part (a) can be written as:

$$V_s - I_{R_1}R_1 - I_{R_2}R_2 = 0$$

$$V_s - I_{R_1}R_1 - I_{R_3}R_3 - I_{R_4}R_4 = 0$$

$$I_{R_2}R_2 - I_{R_3}R_3 - I_{R_4}R_4 = 0$$

Write the KCL and KVL equations in the matrix-vector multiplication form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & R_1 & R_2 & 0 & 0 \\ 0 & R_1 & 0 & R_3 & R_4 \\ 0 & 0 & R_2 & -R_3 & -R_4 \end{bmatrix} \begin{bmatrix} I_s \\ I_{R_1} \\ I_{R_2} \\ I_{R_3} \\ I_{R_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ V_s \\ 0 \end{bmatrix}$$

Plug in the values of the resistors (in kΩ) and the voltage source (in V):

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 0 & 0 & 4 & -2 & -2 \end{bmatrix} \begin{bmatrix} I_s \\ I_{R_1} \\ I_{R_2} \\ I_{R_3} \\ I_{R_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \\ 5 \\ 0 \end{bmatrix}$$

The solution is (in mA)

$$\begin{bmatrix} I_s \\ I_{R_1} \\ I_{R_2} \\ I_{R_3} \\ I_{R_4} \end{bmatrix} = \begin{bmatrix} -5/6 \\ 5/6 \\ 5/12 \\ 5/12 \\ 5/12 \end{bmatrix}$$

Therefore, the voltage between A and B is $V_{R_4} = I_{R_4}R_4 = 5/12\,\text{mA} \cdot 2\,\text{k}\Omega = 5/6\,\text{V} = 0.833\,\text{V}$.

