EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 7A

1. **Proof** *Topics: proof, null space, invertibility.* Consider a square matrix $\mathbf{A}$. Prove that if $\mathbf{A}$ has a non-trivial nullspace, i.e. if the nullspace of $\mathbf{A}$ contains more than just $\vec{0}$, then matrix $\mathbf{A}$ is not invertible.

   We are given that the nullspace of $\mathbf{A}$ contains a vector other than $\vec{0}$. Let such a vector be $\vec{y} \neq \vec{0}$, where $A\vec{y} = \vec{0}$. Imagine, for the sake of contradiction, that $\mathbf{A}$ had an inverse $\mathbf{A}^{-1}$. Then we find that

$$\begin{aligned} & A\vec{y} = \vec{0} \\ \implies \quad & (A^{-1}A)\vec{y} = A^{-1}\vec{0} \\ \implies \quad & \vec{y} = \vec{0}, \end{aligned}$$

   since by the definition of an inverse, $A^{-1}A = \mathbf{I}$.

   But we said that $\vec{y} \neq \vec{0}$, so this is a contradiction! Therefore, our original hypothesis must have been false, so $\mathbf{A}$ cannot have an inverse.

   Thus, the matrix $\mathbf{A}$ is not invertible.

2. **The Romulan Ruse** While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

   (a) *Concept: Matrix Transformations*

   The Romulan illusion technology causes a point (x_0, y_0) to transform or *map* to (u_0, v_0). Similarly, (x_1, y_1) is mapped to (u_1, v_1). Figure 1 and Table 1 show two points on a Romulan ship and the corresponding *mapped* points.

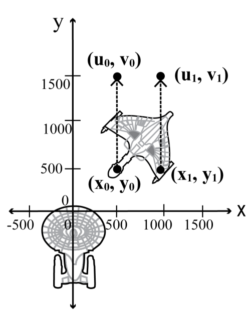

Figure 1: Figure for part (a)

Original Point	Mapped Point
$(x_0, y_0) = (500, 500)$	$(u_0, v_0) = (500, 1500)$

Original Point	Mapped Point
$(x_1, y_1) = (1000, 500)$	$(u_1, v_1) = (1000, 1500)$

Table 1: Original and Mapped Points

   Find a transformation matrix $\mathbf{A}_0$ such that

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

   (b) *Concept: Matrix Transformations*

In this scenario, every point on the Romulan ship (x_m, y_m) is mapped to (u_m, v_m), such that vector $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$ is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

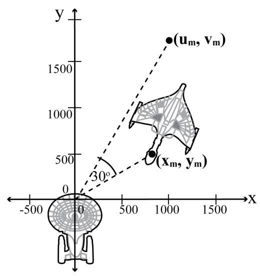

Figure 2: Figure for part (b)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Table 2: Trigononometric Table

Find a transformation matrix R such that $\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_m \\ y_m \end{bmatrix}$.

Transformation matrix that rotates a vector counterclockwise by 30° is:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Transformation matrix that rotates a vector counterclockwise by 30° and scales by 2 is:

$$\mathbf{R} = 2\mathbf{R}_\theta = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Alternatively, the transformation matrix can be written as:

$$\mathbf{R} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point $(0,0)$ towards the probe.

(c) *Concept: Gaussian Elimination, Systems of Equations*

The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix $\mathbf{A}_p$:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

They positioned the probe at (x_p, y_p) so that it maps to $(u_p, v_p) = (0,0)$, where $\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}$. This scenario is shown in Figure 3. The initial position of the torpedo is $(0,0)$ and the torpedo cannot be fired on its initial position! Impressive trick indeed!

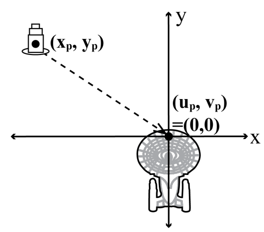

Figure 3: Figure for part (c)

Find the possible positions of the probe (x_p, y_p) so that $(u_p, v_p) = (0,0)$.

We need to solve for

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So essentially we need to find the nullspace of the matrix $\mathbf{A}_p$. Using Gaussian Elimination on the augmented matrix, we have:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_p + 3y_p = 0 \Rightarrow x_p = -3y_p.$$

The solution is $\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where α is $\{\alpha \in \mathbb{R}\}$. So $\begin{bmatrix} x_p \\ y_p \end{bmatrix}$ should be in the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
Alternatively, any point (x_p, y_p) that is on the line: $x = -3y$, would represent all possible positions of the probe.

(d) *Concept: Eigenspaces/Eigenvectors/Eigenvalues*

It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, $\mathbf{A}_p$, mapped it to (u_q, v_q), where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo while traveling along a straight line from $(0,0)$ to (u_q, v_q), hit the probe at (x_q, y_q) on the way!
The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

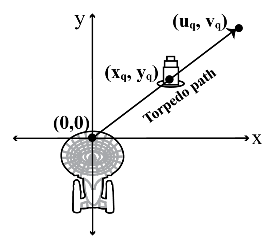

Figure 4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on its initial position $(0,0)$. This means that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$ cannot be $(0,0)$.

We need to solve for $\mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, i.e. we need to find the eigenvectors of $\mathbf{A}_p$. Let's start by finding the eigenvalues:

$$\det\left\{ \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right\} = 0$$
$$\det\left\{ \begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix} \right\} = 0$$

So we have the characteristic polynomial:

$$(1-\lambda)(6-\lambda) - (3)(2) = 0$$
$$\Rightarrow \lambda = 0, 7$$

Using $\lambda = 0$, we have: $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}x_q\\y_q\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ which will map (x_q, y_q) to the original position of the torpedo. The torpedo cannot be fired on its original position. So $\lambda = 0$ will not provide a valid solution.

Using $\lambda = 7$, we have:

$$(\mathbf{A_p} - 7\mathbf{I})\begin{bmatrix}x_q\\y_q\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow= \left(\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix} - 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}x_q\\y_q\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-6 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}x_q\\y_q\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Using Gaussian Elimination on the augmented matrix form, we have

$$\left[\begin{array}{cc|c}-6 & 3 & 0\\2 & -1 & 0\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}2 & -1 & 0\\2 & -1 & 0\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}2 & -1 & 0\\0 & 0 & 0\end{array}\right] \Rightarrow 2x_q - y_q = 0 \Rightarrow y_q = 2x_q$$

The solution is $\alpha\begin{bmatrix}1\\2\end{bmatrix}$, where α is $\{\alpha \in \mathbb{R} : \alpha \neq 0\}$. So $\begin{bmatrix}x_q\\y_q\end{bmatrix}$ should be in the span of $\begin{bmatrix}1\\2\end{bmatrix}$.

Alternatively, any point (x_q, y_q) that is on the line: $y = 2x$, excluding $(0,0)$, would represent all possible positions of the probe.