EECS 16A   Designing Information Devices and Systems I
Fall 2021
Discussion 12B

1. **Inner Product Properties**

   For this question we will verify our coordinate definition of the inner product

   $$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n , \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

   indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

   (a) Show symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$:

   Answer: This is seen by direct expansion:
   Let $x_i, y_i \in \mathbb{R}$, then

   $$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1 \cdot y_1 + x_2 \cdot y_2$$
   $$= y_1 \cdot x_1 + y_2 \cdot x_2$$
   $$= \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle$$

   (b) Show linearity $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c \in \mathbb{R}$ is a real number.

   Answer: This is accomplished through a direct expansion:

   $$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle$$
   $$= x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2)$$
   $$= c(x_1 y_1 + x_2 y_2) + d(x_1 z_1 + x_2 z_2)$$
   $$= c\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + d\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle$$
   $$= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$$

(c) Show non-negativity $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$:

Answer: This part requires just a bit more thought beyond a direct expansion of $\langle \vec{x}, \vec{x} \rangle$, but we first recognize that this inner product is the definition of the norm (or length) of $\vec{x}$. So it is at least in intuitive that a length of some vector (squared) cannot be negative:

$$\langle \vec{x}, \vec{x} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle$$
$$= x_1^2 + x_2^2$$

From this result we notice if either x_1 or x_2 are nonzero (even negative) values, then the inner product HAS to be positive. The only case in which the inner product $\langle \vec{x}, \vec{x} \rangle$ is identically zero is when both $x_1 = 0$ AND $x_2 = 0$, which verifies the final part of the property: $\langle \vec{x}, \vec{x} \rangle = 0$ ONLY IF $\vec{x} = \vec{0}$.

As a bonus, suppose we re-label our vector components $x_1 = a$ and $x_2 = b$.
The we see $\langle \vec{x}, \vec{x} \rangle = c^2 = a^2 + b^2$, which is the Pythagorean theorem!
This verifies that $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = c$ can be geometrically understood as the length of vector $\vec{x}$.

2. **Geometric Interpretation of the Inner Product**
In this problem we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) Derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them. The figure below may be helpful:
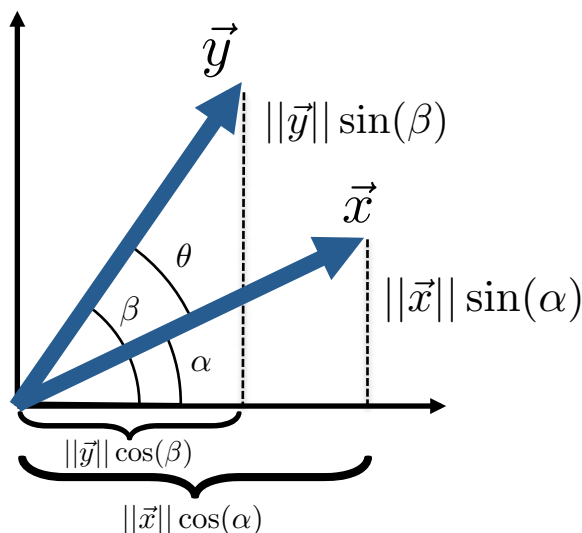
Answer: From trigonometric calculation, if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then we know that $x_1 = \|\vec{x}\| \cdot \cos \alpha$, $x_2 = \|\vec{x}\| \cdot \sin \alpha$, $y_1 = \|\vec{y}\| \cdot \cos \beta$ and $y_2 = \|\vec{y}\| \cdot \sin \beta$ (as in the figure). Then you can directly

write

$$\langle \vec{x}, \vec{y} \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 =$$
$$= \underbrace{\|\vec{x}\| \cdot \cos\alpha}_{x_1} \cdot \underbrace{\|\vec{y}\| \cdot \cos\beta}_{y_1} + \underbrace{\|\vec{x}\| \cdot \sin\alpha}_{x_2} \cdot \underbrace{\|\vec{y}\| \cdot \sin\beta}_{y_2}$$
$$= \|\vec{x}\| \|\vec{y}\| (\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta) =$$
$$= \|\vec{x}\| \|\vec{y}\| \cdot \cos(\beta - \alpha)$$
$$= \|\vec{x}\| \|\vec{y}\| \cdot \cos\theta$$

(b) For each sub-part, identify any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

i. Identify a pair of parallel vectors:

Answer: Parallel vectors point in the same direction (have an angle of $0°$ between them). This means we must have $\vec{y} = \alpha\vec{x}$ for some $\alpha > 0$.
Having only this condition leaves a lot of freedom.
Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = 2\,\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 2 + 1 \cdot 2 = 4$$

ii. Identify a pair of anti-parallel vectors (vectors that point in opposite directions):

Answer: Anti-parallel vectors point in opposite directions (have an angle of $180°$ between them).
This means we must have $\vec{y} = \alpha\vec{x}$ again, but now for some negative $\alpha < 0$.
Having only this condition still leaves a lot of freedom.

Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then set $\vec{y} = -2\,\vec{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot -2 + 1 \cdot -2 = -4$$

iii. Identify a pair of perpendicular vectors:

Answer: Perpendicular vectors point in $90°$ directions with respect to each-other.
Most importantly, the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = 0$ whenever $\vec{x}, \vec{y}$ are perpendicular.
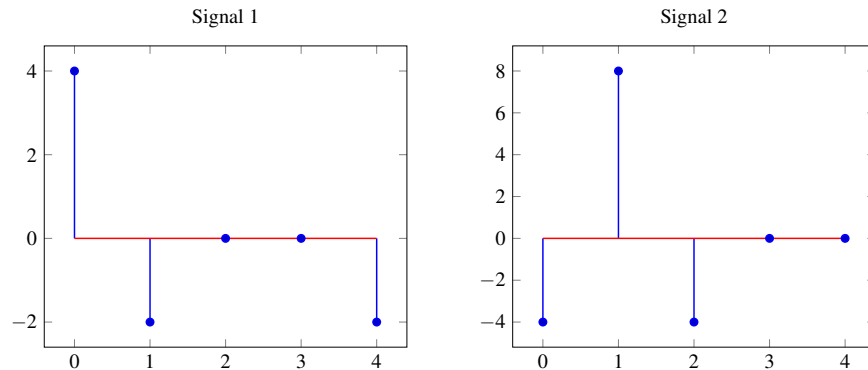
For our example we will fix $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then leave $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ general.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \equiv 0.$$

Thus we must set $y_1 = 0$, but y_2 can assume any nonzero value!

3. Correlation

You are given the following two signals:

(a) Sketch the linear cross-correlation of signal 1 with signal 2, that is find : $\text{corr}(\vec{s}_1, \vec{s}_2)$. Do not assume the signals are periodic.

Answer:

Represent signal 1 as the vector $\vec{s}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & -2 \end{bmatrix}^T$, zero-padded so that we compute only the linear correlation. Similarly, represent signal 2 as the vector $\vec{s}_2 = \begin{bmatrix} -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, where we once again zero pad the vector. Notice we zero pad the front of the vector $\vec{s}_2$ but the back of the vector $\vec{s}_1$.

The cross-correlation between two vectors is defined as follows:
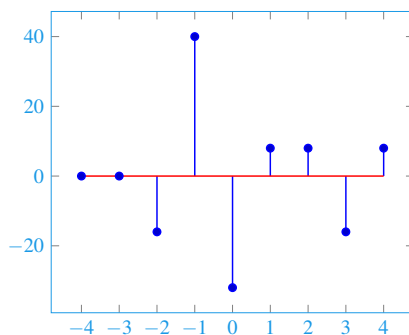
$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$$

To compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$.

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+4]$	-4		8		-4		0		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+4] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+3]$	0		-4		8		-4		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+3] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+2]$	0		0		-4		8		-4		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0	+	0	+	0	+	0	+	-16	+	0	+	0	+	0	+	0	= -16

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+1]$	0		0		0		-4		8		-4		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0	+	0	+	0	+	0	+	32	+	8	+	0	+	0	+	0	= 40

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n]$	0		0		0		0		-4		8		-4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0	+	0	+	0	+	0	+	-16	+	-16	+	0	+	0	+	0	= -32

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-1]$	0		0		0		0		0		-4		-8		-4		0	
$\langle \vec{s}_1, \vec{s}_2[n-1]\rangle$	0	+	0	+	0	+	0	+	0	+	8	+	0	+	0	+	0	= 8

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-2]$	0		0		0		0		0		0		-4		8		-4	
$\langle \vec{s}_1, \vec{s}_2[n-2]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	8	= 8

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-3]$	0		0		0		0		0		0		0		-4		8	
$\langle \vec{s}_1, \vec{s}_2[n-3]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-16	= -16

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-4]$	0		0		0		0		0		0		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-4]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	8	= 8

Non-periodic Cross-correlation of Signals 1 and 2

(b) Assume signal $\vec{s}_2$ is periodic with period 5. Find the linear cross correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$ of the two signals.

Answer: Recall that $\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$

Since signal $\vec{s}_2$ is periodic, it continues on to $+\infty$ and $-\infty$. We'll start by just performing the linear correlation over one period and ignoring the signals outside of this period. Shifting the signal back will bring the next period of the signal into our range of interest. Thus we calculate the linear cross-correlation assuming the signal is periodic as below:

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n]$	−4		8		−4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n]\rangle$	−16	+	−16	+	0	+	0	+	0	= −32

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-1]$	0		−4		8		−4		0	
$\langle \vec{s}_1, \vec{s}_2[n-1]\rangle$	0	+	8	+	0	+	0	+	0	= 8

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-2]$	0		0		−4		8		−4	
$\langle \vec{s}_1, \vec{s}_2[n-2]\rangle$	0	+	0	+	0	+	0	+	8	= 8

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-3]$	−4		0		0		−4		8	
$\langle \vec{s}_1, \vec{s}_2[n-3]\rangle$	−16	+	0	+	0	+	0	+	−16	= −32

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-4]$	8		-4		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$	32	$+$	8	$+$	0	$+$	0	$+$	8	$= 48$

Let's continue to calculate the values of the inner product with more shifts.

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-5]$	-4		8		-4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n-5] \rangle$	-16	$+$	-16	$+$	0	$+$	0	$+$	0	$= -32$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-6]$	0		-4		8		-4		0	
$\langle \vec{s}_1, \vec{s}_2[n-6] \rangle$	0	$+$	8	$+$	0	$+$	0	$+$	0	$= 8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-7]$	0		0		-4		8		-4	
$\langle \vec{s}_1, \vec{s}_2[n-7] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	8	$= 8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-8]$	-4		0		0		-4		8	
$\langle \vec{s}_1, \vec{s}_2[n-8] \rangle$	-16	$+$	0	$+$	0	$+$	0	$+$	-16	$= -32$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-9]$	8		-4		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-9] \rangle$	32	$+$	8	$+$	0	$+$	0	$+$	8	$= 48$

Periodic Cross-correlation of Signals 1 and 2
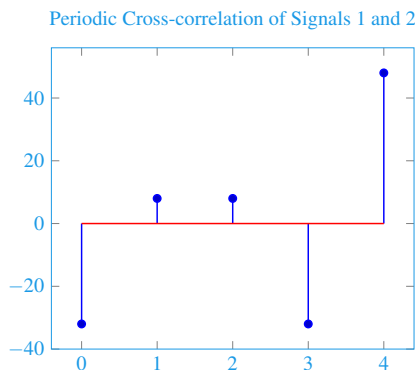
Notice that the pattern repeats.