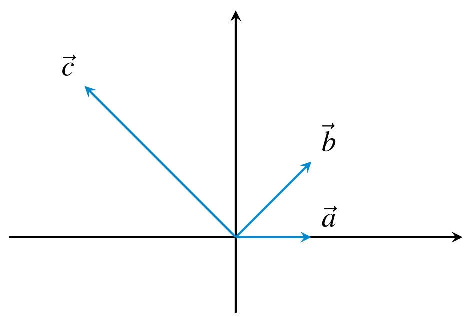
EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 3A

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

(c) Solve for α, β.

2. Span basics

(a) What is span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

(b) Is $\begin{bmatrix}5\\5\\0\end{bmatrix}$ in span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

(c) What is a possible choice for $\vec{v}$ that would make span$\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v}\right\} = \mathbb{R}^3$?

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

3. Proofs

Definition: A set of vectors $\{\vec{v_1}, \vec{v_2}, \dots \vec{v_n}\}$ is **linearly dependent** if there exists constants c_1, c_2, $\dots$ c_n such that $\sum_{i=1}^{i=n} c_i \vec{v_i} = \vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.