EECS 16A Designing Information Devices and Systems I
Fall 2021 Discussion 3B

1. Inverses

In general, the *inverse* of a matrix "undoes" the operation that a matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}\,,$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(k\mathbf{A})^{-1} = \tfrac{1}{k}\mathbf{A}^{-1} \qquad \text{for a nonzero scalar } k \in \mathbb{R}$$

$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T} \qquad T \text{ is "Transpose"}$$

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{assuming } \mathbf{A},\mathbf{B} \text{ are both invertible}$$

(a) Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible matrices.
Prove that $(\mathbf{A}\mathbf{B}\mathbf{C})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$?
(*Bonus:* How does this relate to the invertibility of $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$?)
v. Are all square matrices invertible?

vi. (PRACTICE) How can you find the inverse of a general $n \times n$ matrix?

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Suggest some methods to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by 60°? Your TA will show you the result in the iPython notebook.

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by 60°. Your TA will show you the result in the iPython notebook.

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities!*)

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

(e) Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

(f) What are the matrices that reflect a vector about the (i) x-axis, (ii) y-axis, and (iii) $x = y$

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Your TA will demonstrate parts (a) and (b) in the iPython notebook.

(a) Let's see what happens to the unit square when we rotate the square by $60°$ and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by $60°$. Is this the same as in part (a)?

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

Part 3: Distributivity of Operations

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.

