EECS 16A Dis 12B Notes

Tuesday, November 9, 2021 5:40 PM

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1. Inner Product Properties

For this question we will verify our coordinate definition of the inner product

inner product → $\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$

indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

$\langle x,y \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$ $\quad R^n$

Euclidean inner product/ dot product

$= x^Ty$

(a) Show symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$: "commutative"

$\in R^2$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\langle x,y \rangle = x^Ty = x_1y_1 + x_2y_2 + \ldots + x_ny_n$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $= x_1y_1 + x_2y_2$

scalars

$= y_1x_1 + y_2x_2$

$= \boxed{\langle y,x \rangle}$

$\boxed{\langle x,y \rangle = \langle y,x \rangle}$ ✓

$A \cdot B \neq B \cdot A$

Inner Product properties:

1. Symmetry ✓
2. Linearity ✓
3. Non-negativity ✓

(b) Show linearity $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c \in \mathbb{R}$ is a real number.

- vector addition
- Scalar multiplication

$\langle x, cy + dz \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle$

$= x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2)$

$= cx_1y_1 + cx_2y_2 + dx_1z_1 + dx_2z_2$

$= c(x_1y_1 + x_2y_2) + d(x_1z_1 + x_2z_2)$

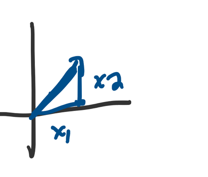
$= c\langle x,y \rangle + d\langle x,z \rangle$ ✓

(c) Show non-negativity $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$:

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$

$=0 \quad =0$

$\geq 0 \quad \geq 0$



$= \langle x,x \rangle \geq 0$

if $x_1 = x_2 = 0$ then $\langle x,x \rangle = 0$

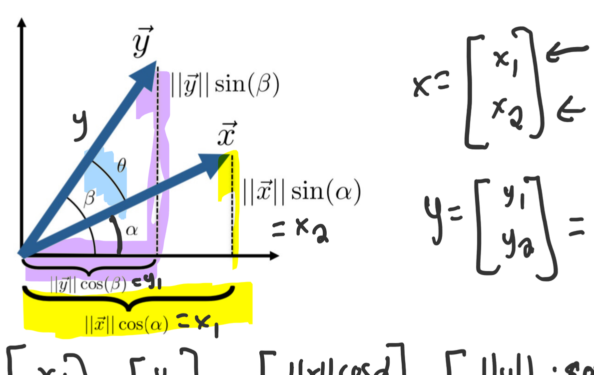
$\langle x,x \rangle = ||x||^2$ $\quad ||x|| = \sqrt{\langle x,x \rangle}$

length of x

2. Geometric Interpretation of the Inner Product

In this problem we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) Derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them. The figure below may be helpful:



$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ← x-component, ← y-component $= \begin{bmatrix} ||x|| \cdot \cos\alpha \\ ||x|| \cdot \sin\alpha \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ||y|| \cdot \cos\beta \\ ||y|| \cdot \sin\beta \end{bmatrix}$

$\langle x,y \rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ||x||\cos\alpha \\ ||x||\sin\beta \end{bmatrix} \cdot \begin{bmatrix} ||y|| \cdot \cos\beta \\ ||y|| \cdot \sin\beta \end{bmatrix}$

$= ||x||\,||y|| \cdot \cos\alpha\cos\beta + ||x||\,||y|| \cdot \sin\alpha\sin\beta$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \cos(B-A)$

$= ||x||\,||y|| \cdot \cos(\beta - \alpha)$ $\quad = \cos(\alpha - \beta)$ $\quad \cos(-\beta) = \cos(\beta)$

$= \cos(\beta - \alpha)$

$= \boxed{\underbrace{||x|| \cdot ||y||}_{\text{magnitudes}} \cdot \cos(\theta)}$

angle difference

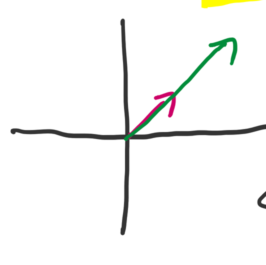
(b) For each sub-part, identify any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

i. Identify a pair of parallel vectors:

Point in same direction

$x = d \cdot y$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

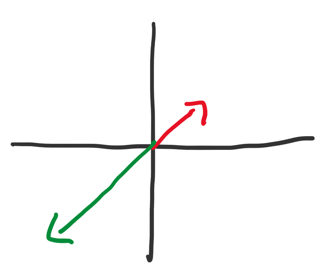


$\langle x,y \rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 + 2 = \boxed{4}$

$\theta = 0°$

ii. Identify a pair of anti-parallel vectors (vectors that point in opposite directions):

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$



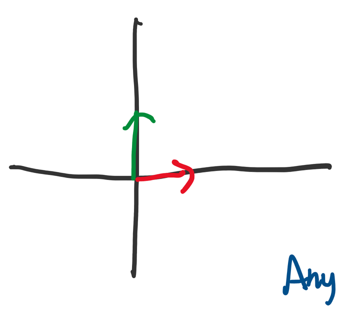
$\langle x,y \rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \boxed{-4}$

$= ||x||\,||y|| \cdot \cos\theta$

$\theta = 180°$

iii. Identify a pair of perpendicular vectors:

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\langle x,y \rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{0}$

Any orthogonal vectors have inner product of 0 ☆

Inner Products:

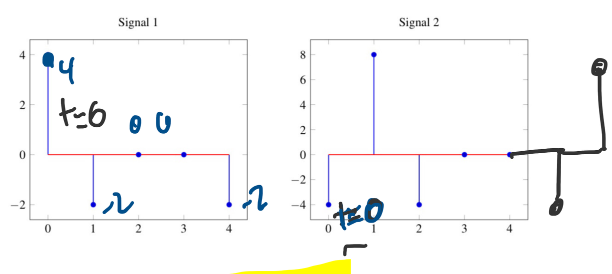
$x^Ty \in R^n$

$n \times 1 \quad n \times 1$

$(1 \times n) \cdot (n \times 1) = \boxed{1 \times 1}$

3. Correlation

You are given the following two signals:



Inner Product

x [ ]

y [ ]

Cross-Correlation: shifted inner products

(a) Sketch the linear cross-correlation of signal 1 with signal 2, that is find : corr($\vec{s}_1, \vec{s}_2$). Do not assume the signals are periodic.

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n+4]$			-4	0	0				
$\langle \vec{s}_1, \vec{s}_2[n+4] \rangle$									

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n+3]$				-4	0				
$\langle \vec{s}_1, \vec{s}_2[n+3] \rangle$									

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n+2]$			-4	8	-4				
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$									

t=0

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2	
$\vec{s}_2[n+1]$ (n=0)	0	0	0	-4	8 ($s_2[1]$)	-4	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$					32+8					$= \boxed{40}$

☆ left shift by 1

t=0

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2	
$\vec{s}_2[n]$	0	0	0	0	-4	8	4	0	0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0+0+0+0 -16 -16 +0+0 +0									$= \boxed{-32}$

☆ no-shift

t=0

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2	
$\vec{s}_2[n-1]$	0 0	0	0	0 (t=-1)	-4	8	-4	0		
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$					8					$= \boxed{8}$

☆ n-1 right shift by 1

[_ _ _ _ _]

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n-2]$									
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$									

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n-3]$									
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$									

$\vec{s}_1$	0	0	0	0	4	-2	0	0	-2
$\vec{s}_2[n-4]$									-4
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$									

(b) Assume signal $\vec{s}_2$ is periodic with period 5. Find the linear cross correlation corr($\vec{s}_1, \vec{s}_2$) of the two signals.

$\vec{s}_1$	4	-2	0	0	-2	
$\vec{s}_2[n]$	-4	8	4	0	0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	16+16					$= \boxed{32}$

no shift

$\vec{s}_1$	4	-2	0	0	-2
$\vec{s}_2[n-1]$	0	-4	8	4	0
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$					

right shift by 1

$\vec{s}_1$	4	-2	0	0	-2
$\vec{s}_2[n-2]$	0	0	-4	8	4
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$					

$\vec{s}_1$	4	-2	0	0	-2
$\vec{s}_2[n-3]$	4	0	0	-4	8
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$					

$\vec{s}_1$	4	-2	0	0	-2
$\vec{s}_2[n-4]$	8	4	0	0	-4
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$					

right shifted by 4

-4 8 4 0 0 $\quad$ n-5 $\quad$ right shift by 5