# EECS 16A Dis 14B Notes

Tuesday, November 30, 2021 6:16 PM

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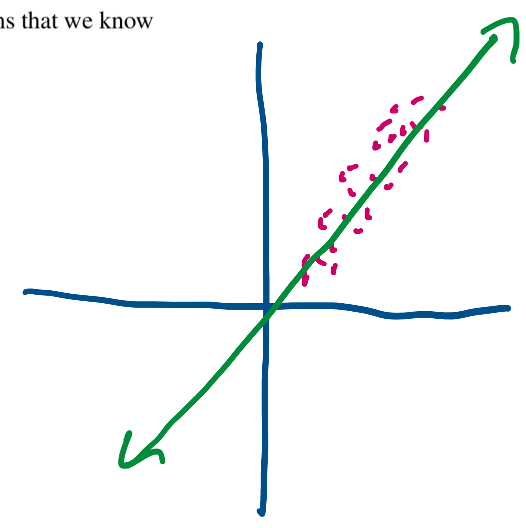
**1. Polynomial Fitting**

Let's try an example. Say we know that the output, $y$, is a quartic polynomial in $x$. This means that we know that $y$ and $x$ are related as follows:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$



We're also given the following observations:

| $x$ | $y$ |
|---|---|
| 0.0 | 24.0 |
| 0.5 | 6.61 |
| 1.0 | 0.0 |
| 1.5 | −0.95 |
| 2.0 | 0.07 |
| 2.5 | 0.73 |
| 3.0 | −0.12 |
| 3.5 | −0.83 |
| 4.0 | −0.04 |
| 4.5 | 6.42 |

(a) What are the unknowns in this question?=

$$\underset{\text{knowns}}{y} = a_0 + a_1 \underset{\text{knowns}}{x} + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Unknowns: $a_0, a_1, a_2, a_3, a_4$

(b) Can you write an equation corresponding to the first observation $(x_0, y_0)$, in terms of $a_0, a_1, a_2, a_3$, and $a_4$? What does this equation look like? Is it linear in the unknowns?

$(0.0, 24.0)$

$y = 24.0 = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + a_4 \cdot 0^4$

$24 = a_0$

(c) Now, write a system of equations in terms of $a_0, a_1, a_2, a_3$, and $a_4$ using *all of the observations*.

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 \\ \vdots & & & & \\ 1 & x_9 & x_9^2 & x_9^3 & x_9^4 \end{bmatrix}}_{\text{Data matrix } (10,5)} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\text{unknowns } (5,1)} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_9 \end{bmatrix}}_{\text{measurements } (10,1)}$$

$D \cdot \vec{x} = \vec{y}$

Least squares!

If unique sol, $\hat{\vec{x}} = \vec{x}$

(d) Finally, solve for $a_0, a_1, a_2, a_3$, and $a_4$ using IPython or any method you like. You have now found the quartic polynomial that best fits the data!

See iPython Notebook

**2. Orthogonal Subspaces**

Two vectors are $\vec{x}$ and $\vec{y}$ are said to be orthogonal if their inner product is zero. That is $\langle \vec{x}, \vec{y} \rangle = 0$. $x^T \cdot y = 0$

Two subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{R}^N$ are said to be orthogonal if all vectors in $\mathbb{S}_1$ are orthogonal to all vectors in $\mathbb{S}_2$. That is,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = 0 \; \forall \vec{v}_1 \in \mathbb{S}_1, \vec{v}_2 \in \mathbb{S}_2.$$

(a) Recall that the *column space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the columns of $\mathbf{A}$ and that the *null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\mathbf{A}\vec{v} = \vec{0}$.

Prove that for any matrix $\mathbf{A}$, the column space of $\mathbf{A}^T$ and null space of $\mathbf{A}$ are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}^T) \perp \text{Null}(\mathbf{A}) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.

Hint: Use the row interpretation of matrix multiplication.

Know: $\text{Col}(A^T) = \text{span}\{\vec{b} \mid A^T \cdot \vec{x} = \vec{b}\}$

$\text{Null}(A) = \text{span}\{\vec{v} \mid A \cdot \vec{v} = \vec{0}\}$

Show: $\langle b, v \rangle = 0$ $\quad b^T \cdot v = 0$

$b^T v = (A^T \cdot x)^T \cdot v = x^T \cdot A^{T^T} \cdot v$

$b = A^T \cdot x$

$= x^T \cdot A \cdot v$

$= x^T \cdot 0$

$= 0$

$\langle b, v \rangle = 0$

$\text{Col}(A^T) \perp \text{Null}(A)$

Row interpretation of matrix-vector multiplication

$$A^T = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_m \\ | & & | \end{bmatrix}$$

$A\vec{v} = 0$

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} = \begin{bmatrix} — s_1^T — \\ — s_2^T — \\ \vdots \\ — s_m^T — \end{bmatrix} \cdot v = \begin{bmatrix} s_1^T v \\ s_2^T \cdot v \\ \vdots \\ s_m^T \cdot v \end{bmatrix} = \begin{bmatrix} \langle s_1, v \rangle \\ \vdots \\ \langle s_m, v \rangle \end{bmatrix} = \vec{0}$$

$A \cdot v = 0$

$v \in \text{Null}(A)$

$\text{Col}(A^T) = \text{span}\{s_1, \ldots, s_m\}$ any vector $\in \text{Col}(A^T)$ is a linear combo of $s_1, \ldots, s_m$

$\text{Col}(A^T) \perp \text{Null}(A)$

(b) Now prove that for any matrix $\mathbf{A}$, the column space and null space of $\mathbf{A}^T$ are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}) \perp \text{Null}(\mathbf{A}^T) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.

Know: $\text{Col}(A^T) \perp \text{Null}(A)$

Show: $\text{Col}(A) \perp \text{Null}(A^T)$

let $B = A^T \quad B^T = A^{T^T} = A$

$\text{Col}(B^T) \perp \text{Null}(B)$

$$\boxed{\text{Col}(A) \perp \text{Null}(A^T)}$$