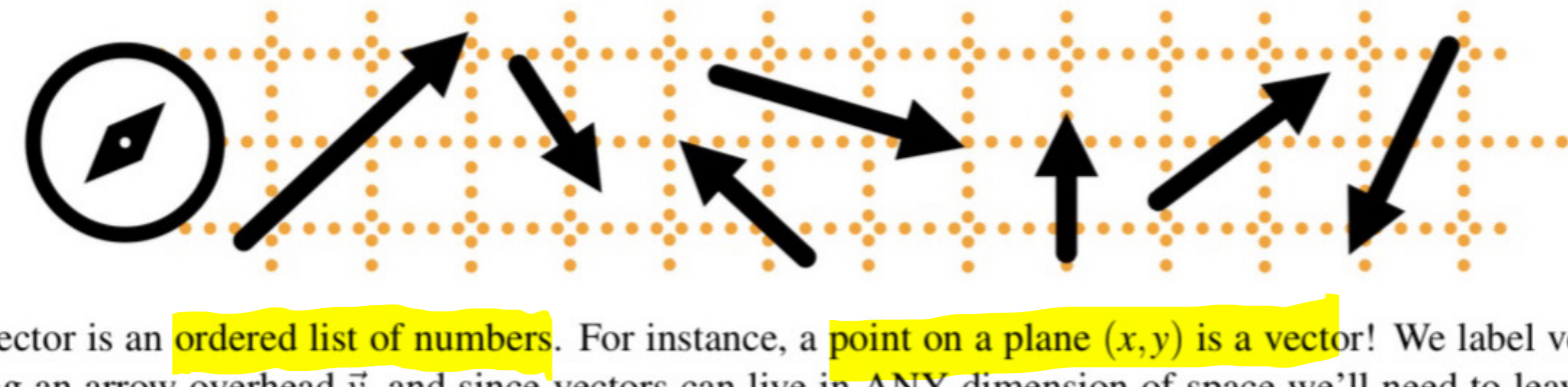


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1. Vectors



← indicates vector

A vector is an **ordered list of numbers**. For instance, a **point on a plane (x, y)** is a vector! We label vectors using an arrow overhead \vec{v} , and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x, y) \rightarrow \vec{v} = (v_1, v_2, \dots)$. Below are a few more examples (the left-most form is the general definition):

column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

rows column

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftarrow \begin{matrix} x \text{ component} \\ y \text{ component} \end{matrix}$$

Just to unpack this a bit more, $\mathbb{R} \in \mathbb{R}^1$ in english means "vector \mathbb{R} lives in 3-Dimensional space".

- The \in symbol literally means "in"
- The \mathbb{R} stands for "real numbers" (FUN FACT: \mathbb{Z} means "integers" like $-2, -4, 0, \dots$)
- The exponent \mathbb{R}^n indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{c}$ above. We call these **column vectors**, in contrast to horizontally written vectors which we call **row vectors**.

Okay, let's dig into a few examples:

(a) Which of the following vectors live in \mathbb{R}^2 space?

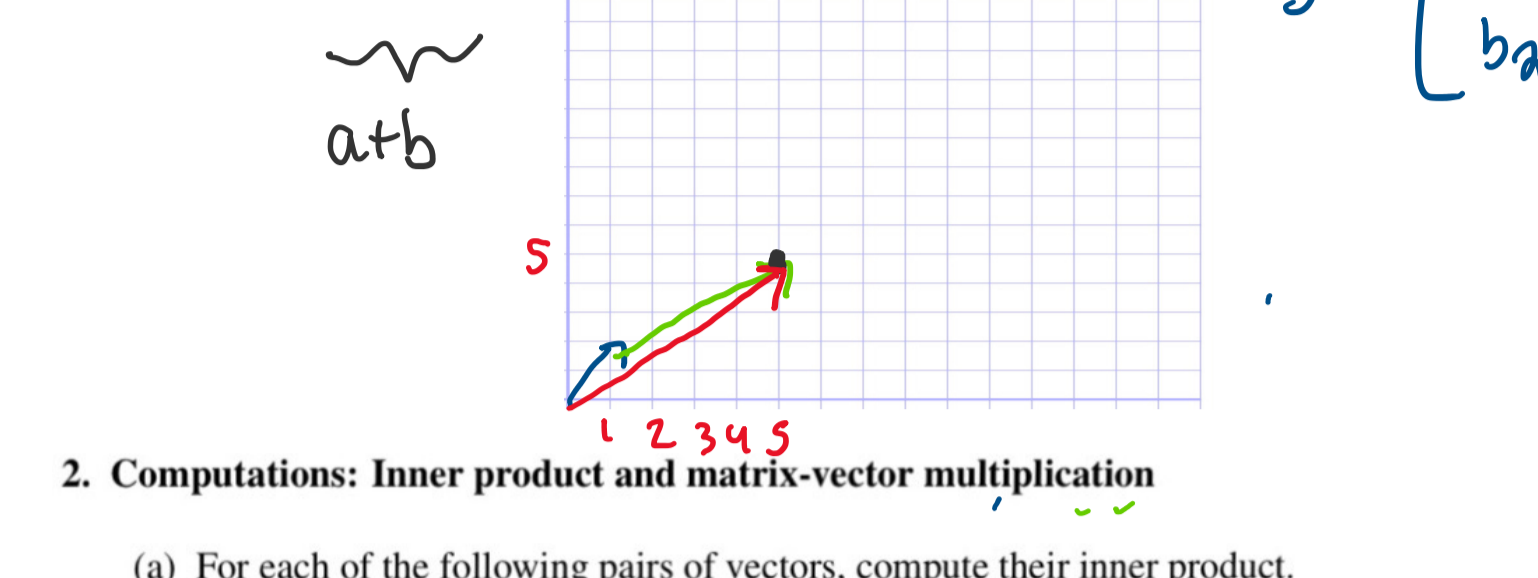
i. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ (circled in red)

ii. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ (circled in red)

iii. $\begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \in \mathbb{R}^3$ (circled in red)

iv. $\begin{bmatrix} -20 \\ 100 \end{bmatrix} \in \mathbb{R}^2$ (circled in green)

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):



(c) Compute the sum $\vec{a} + \vec{b} = \vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \vec{c} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\vec{a} + \vec{b} = \vec{c}$$

2. Computations: Inner product and matrix-vector multiplication

(a) For each of the following pairs of vectors, compute their inner product.

i. $\vec{a} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ 6 \\ 1 \end{bmatrix}$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = 1(-6) + 6(6) + 1(1) = -6 + 36 + 1 = 31$$

ii. $\vec{a} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \end{bmatrix}$

$$\vec{a} \cdot \vec{b} = 2(6) + 6(2) + 12(-1) + (-4)(3) = 12 + 12 - 12 - 12 = 0$$

(b) Perform matrix vector multiplication to compute $A\vec{b}$ in each of the following cases:

i. $A = \begin{bmatrix} 1 & 6 \\ 2 & -7 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A\vec{b} = \begin{bmatrix} 1(1) + 6(2) \\ 2(1) + (-7)(2) \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

ii. $A = \begin{bmatrix} 1 & 9 & 2 \\ 7 & 10 & -7 \\ -1 & 2 & -8 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$$A\vec{b} = \begin{bmatrix} 1(1) + 9(0) + 2(3) \\ 7(1) + 10(0) + (-7)(3) \\ -1(1) + 2(0) + (-8)(3) \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \\ -25 \end{bmatrix}$$

3. Mini-Lecture: Matrix-Vector Form for Systems of Linear Equations

Consider a system of linear equations with 3 unknowns x_1, x_2, x_3 . As we have seen, such a system of equations can be written in the general form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \quad (1)$$

where the a_{ij} and b_i are all real-valued constants.

We previously introduced the augmented matrix notation, which was useful for systematically determining solutions (e.g., using Gaussian elimination). Now, using matrix-vector multiplication, observe that we can also write this system in **matrix-vector form**:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (2)$$

If we denote the coefficient matrix by A , the variable vector by \vec{x} and the vector of constant terms by \vec{b} then the system of equations can be concisely written as

$$A\vec{x} = \vec{b} \quad (3)$$

This is called the **matrix-vector form** (or **matrix-vector representation**) of the system of linear equations Eq. (1).

Now that we understand how to multiply matrices with vectors, we can revisit this representation of systems of equations, to demonstrate that it is algebraically equivalent to the original system.

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (4)$$

Notice that the left-hand-sides of our original system of equations have suddenly appeared as components of the vector on the left-hand-side! Since equating two vectors is the same as equating their corresponding coefficients, we can rewrite the above vector equation as a set of n scalar equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

which are exactly the equations in our linear system.

Gaussian elimination:

1. Create an augmented matrix

$$\begin{array}{cc|c} (1) & x_1 + x_2 & 4 \\ & x_1 + 2x_2 & 7 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 7 \end{bmatrix} \leftarrow \text{implies there are 2 unknowns}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix} \leftarrow \text{solutions}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{cases} x_1 + x_2 = 4 \\ x_1 + 2x_2 = 7 \end{cases}$$

4. Matrix Multiplication

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}, F = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}, G = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, H = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a) AB

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(3) + 4(2) \\ 2(3) + 3(2) \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$$

(b) CD

$$C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1(3) + 4(2) & 1(2) + 4(1) \\ 2(3) + 3(2) & 2(2) + 3(1) \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}$$

(c) DC

$$D = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$DC = \begin{bmatrix} 3(1) + 2(2) & 3(4) + 2(3) \\ 2(1) + 1(2) & 2(4) + 1(3) \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}$$

(d) CE

$$C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, E = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

$$CE = \begin{bmatrix} 1(1) + 4(4) & 1(9) + 4(3) & 1(5) + 4(2) & 1(7) + 4(2) \\ 2(1) + 3(4) & 2(9) + 3(3) & 2(5) + 3(2) & 2(7) + 3(2) \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}$$

(e) FE (only note whether or not the product exists)

$$F = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

No product (4 columns in F ≠ 4 rows in E)

(f) EF (only note whether or not the product exists)

$$E = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}, F = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}$$

Product exists (4 columns in E = 4 rows in F)

(g) GH

$$G = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, H = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

$$GH = \begin{bmatrix} 8(5) + 1(1) + 6(2) & 8(3) + 1(8) + 6(3) & 8(4) + 1(2) + 6(5) \\ 3(5) + 5(1) + 7(2) & 3(3) + 5(8) + 7(3) & 3(4) + 5(2) + 7(5) \\ 4(5) + 9(1) + 2(2) & 4(3) + 9(8) + 2(3) & 4(4) + 9(2) + 2(5) \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}$$

(h) HG

$$H = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}, G = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

$$HG = \begin{bmatrix} 5(8) + 3(3) + 4(4) & 5(1) + 3(8) + 4(9) & 5(6) + 3(7) + 4(2) \\ 1(8) + 8(3) + 2(4) & 1(1) + 8(8) + 2(9) & 1(6) + 8(7) + 2(2) \\ 2(8) + 3(3) + 5(4) & 2(1) + 3(8) + 5(9) & 2(6) + 3(7) + 5(2) \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}$$