EECS 16A Dis 3A Notes

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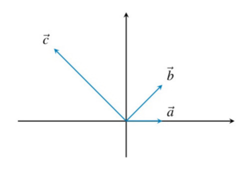
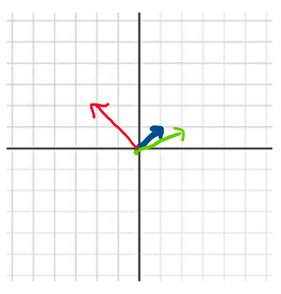
1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.



(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

$\alpha a + \beta b = c \longrightarrow \alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$

$\alpha x + \beta y = z$

$\begin{bmatrix}\alpha+2\beta\\\alpha+\beta\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix} \longleftrightarrow \boxed{\left[\begin{array}{cc|c}1&2&-2\\1&1&2\end{array}\right]}$

(c) Solve for α,β.

1. $\left[\begin{array}{cc|c}1&2&-2\\1&1&2\end{array}\right]$

2. $-R_1+R_2=R_2$ $\quad\left[\begin{array}{cc|c}1&2&-2\\0&-1&4\end{array}\right]$ $\quad R_2=-R_2$ $\quad\left[\begin{array}{cc|c}1&2&-2\\0&1&-4\end{array}\right]$

3. $R_2=-2R_1+R_2$ $\quad\left[\begin{array}{cc|c}1&0&6\\0&1&-4\end{array}\right]$ $\quad\boxed{\begin{bmatrix}6\\-4\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\end{bmatrix}}$

2. Span basics

(a) What is span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

$\mathbb{R}^3$

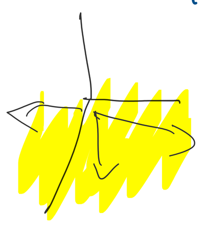
$\alpha_i \in \mathbb{R}$

$\boxed{\text{Not } \mathbb{R}^3}$

$\begin{bmatrix}1\\1\\0\end{bmatrix}$

$\alpha_1\begin{bmatrix}1\\2\\0\end{bmatrix}+\alpha_2\begin{bmatrix}2\\1\\0\end{bmatrix}$

$=\begin{bmatrix}\alpha_1+2\alpha_2\\2\alpha_1+\alpha_2\\0\end{bmatrix}=\boxed{\begin{bmatrix}*\\*\\0\end{bmatrix}}$ $\quad\begin{bmatrix}5\\1\\0\end{bmatrix}$ $\quad\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad\begin{bmatrix}*\\*\\*\end{bmatrix}$



(b) Is $\begin{bmatrix}5\\5\\0\end{bmatrix}$ in span $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$?

$\begin{bmatrix}*\\*\\0\end{bmatrix}$ $\quad\left[\begin{array}{cc|c}1&2&5\\2&1&5\\0&0&0\end{array}\right]$ $\quad\alpha_1=\frac{5}{3}\quad\alpha_2=\frac{5}{3}$

$\boxed{\text{Yes}}$

(c) What is a possible choice for $\vec{v}$ that would make span$\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix},\vec{v}\right\}=\mathbb{R}^3$?

$\begin{bmatrix}*\\*\\0\end{bmatrix}$ $\quad\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad\begin{bmatrix}1600\\500\\-99\end{bmatrix}$

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\begin{bmatrix}1&2\\2&1\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

unique sol

∞ sol

$\left[\begin{array}{cc|c}1&2&b_1\\2&1&b_2\\0&0&b_3\end{array}\right] \longrightarrow \begin{bmatrix}*\\*\\0\end{bmatrix}$

$\boxed{0=b_3}$

3. Proofs

Definition: A set of vectors $\{\vec{v}_1,\vec{v}_2,\dots\vec{v}_n\}$ is **linearly dependent** if there exists constants $c_1, c_2, \dots\ c_n$ such that $\sum_{i=1}^{i=n} c_i\vec{v}_i=\vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

$c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_n\vec{v}_n=\vec{0}$

$c_i=0$

Ex. $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}-1\\0\end{bmatrix}$

$\alpha_1\begin{bmatrix}1\\0\end{bmatrix}+\alpha_2\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\alpha_1=1\quad\alpha_2=1$

$\alpha_1=2\quad\alpha_2=2$

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x}=\vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

$\begin{bmatrix}|&&|\\\vec{a}_1&\cdots&\vec{a}_n\\|&&|\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=0$

$x_1\vec{a}_1+x_2\vec{a}_2+\dots+x_n\vec{a}_n=0$ $\quad$ some $x_i\neq0$

$\boxed{\sum_{i=1}^{n}x_i\vec{a}_i=0} \longleftarrow$ A's cols L.D.

(b) For $\mathbf{A}\in\mathbb{R}^{m\times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x}=\vec{b}$, that is, $\mathbf{A}\vec{x}_1=\vec{b}$ and $\mathbf{A}\vec{x}_2=\vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Know & show

State your knowns $\longrightarrow$ show your solution

1. $A\vec{x}_1=\vec{b}\quad A\vec{x}_2=b$	A's cols L.D
$A\vec{x}_1-A\vec{x}_2=\vec{b}-\vec{b}=\vec{0}$	$A\vec{x}=\vec{0}$
$A\vec{x}_1-A\vec{x}_2=\vec{0}$	
$A(\vec{x}_1-\vec{x}_2)=0$	

$x_1\neq x_2\quad x_1-x_2\neq0 = \vec{x}_3$ $\quad\boxed{A\vec{x}_3=\vec{0}}$

(c) Let $\mathbf{A}\in\mathbb{R}^{m\times n}$ be a matrix for which there exists a non-zero $\vec{y}\in\mathbb{R}^n$ such that $\mathbf{A}\vec{y}=\vec{0}$. Let $\vec{b}\in\mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x}=\vec{b}$, then there are infinitely many solutions.

Knowns

$A\vec{x}=b\quad A\vec{y}=0$

$A\vec{x}+\vec{0}=\vec{b}+\vec{0}$

$A\vec{x}+A\vec{y}=\vec{b}$

$A(\vec{x}+\vec{y})=\vec{b}$

$\vec{x}+\vec{y}=z$

$\boxed{A\vec{z}=\vec{b}}$

$\left\{\begin{array}{l}A\vec{x}_1+A\vec{x}_2=\vec{0}\\A(\vec{x}_1+\vec{x}_2)=\vec{0}\end{array}\right.$

$5\cdot3+5\cdot2$

$5(3+2)$

$A(x_1+x_2)$

$AB+AC$

$A(B+C)\neq(B+C)A$

$(x_1$

$Ax_1+Ax_2\neq(x_1+x_2)A$

$z=\vec{x}+\vec{y}$

$=\vec{x}+\vec{0}$

$=\vec{x}+cy\quad c\in\mathbb{R}$

$A(\vec{x}+c\vec{y})=\vec{b}\longrightarrow\infty$ sol!

Linear combination: $\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\dots+\alpha_n\vec{v}_n=\vec{b}$

Sum of weighted vectors: $\sum_{i=1}^{n}\alpha_i\vec{v}_i$

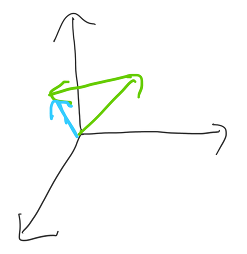
b is a lin. combo of our vectors $v_1\dots v_n$

Span: Set of all possible linear combinations of $\{\vec{v}_1,\dots,\vec{v}_n\}$

$\text{span}(v_1,\dots v_n)=\left\{\underbrace{\sum_{i=1}^{n}\alpha_i\vec{v}_i}_{\text{linear combinations}}\ \middle|\ \alpha_i\in\mathbb{R}\right\}$

"reachable vectors"

$\{\alpha_1v_1+\alpha_2v_2+\dots+\alpha_nv_n\}$



b is a lin. combo of A's columns $Ax=b$

$b\in\text{col}(A)$

1) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ $\begin{bmatrix}2\\1\end{bmatrix}$

$\alpha_1\begin{bmatrix}1\\0\end{bmatrix}+\alpha_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

$\underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{A}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

$\left[\begin{array}{cc|c}1&1&2\\0&1&1\end{array}\right]$ $\quad\begin{bmatrix}2\\1\end{bmatrix}$ is a lin. combo of A's cols

$\text{Span}\left\{\underbrace{\begin{bmatrix}1\\0\end{bmatrix}\ \&\ \begin{bmatrix}1\\1\end{bmatrix}}_{\text{linearly independent}}\right\}=\mathbb{R}^2$

$\left[\begin{array}{cc|c}1&2&3\\0&0&0\end{array}\right]$ $\quad\begin{bmatrix}*\\0\end{bmatrix}$

$\begin{bmatrix}\alpha_1+\alpha_2\\2\alpha_1+2\alpha_2\\0\end{bmatrix}=\begin{bmatrix}\alpha_1+\alpha_2\\0\\0\end{bmatrix}+2\begin{bmatrix}0\\\alpha_1+\alpha_2\\0\end{bmatrix}$ $\quad\alpha_1=500\quad\alpha_2=5\cdot\alpha_1$ $\quad\begin{bmatrix}*\\*\\0\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\end{bmatrix}+2\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\2\\0\end{bmatrix}$

Questions:

1. How do we know our span is the x-y plane given vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$? (Prob 2a)

$\alpha_1\begin{bmatrix}1\\2\\0\end{bmatrix}+\alpha_2\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}\alpha_1+2\alpha_2\\2\alpha_1+\alpha_2\\0\end{bmatrix}=\begin{bmatrix}*\\*\\0\end{bmatrix}$

* means any value (can be 0)

From 2d,

$\left[\begin{array}{cc|c}1&2&b_1\\2&1&b_2\\0&0&b_3\end{array}\right]$

The system is consistent (has ≥1 solution) as long as $b_3=0$. Thus b_1 & b_2 can be any value $\longrightarrow\begin{bmatrix}*\\*\\0\end{bmatrix}$

The span of the vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ is not all of $\mathbb{R}^3$ but rather the x-y plane

2. If $\vec{b}\in\text{col}(A)$, does that mean A has linearly independent columns? (Prob 1c)

In prob 1c, the cols of A are LI but just because $b\in\text{col}(A)$ doesn't always mean A's cols are LI

Ex. $\left[\begin{array}{cc|c}1&2&3\\0&0&0\end{array}\right]$ where $A=\begin{bmatrix}1&2\\0&0\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}3\\0\end{bmatrix}$

$\vec{b}\in\text{col}(A)$ but there are ∞ many sol!

For $A\vec{x}=\vec{b}$:

1. $\vec{b}\in\text{col}(A)\longrightarrow$ system is consistent
   1. A's cols are L.D $\longrightarrow\infty$ sol
   2. A's cols are L.I. $\longrightarrow$ unique sol
2. $\vec{b}\notin\text{col}(A)\longrightarrow$ system is inconsistent (no sol)