EECS 16A Dis 4B Notes

Monday, September 20, 2021

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Recall from lecture the way to compute a determinant of any 2×2 matrix is by using the following formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$

$\text{Det} = 0 \iff$ L.D. columns
A is not invertible
A has a non-trivial nullspace

1. Mechanical Determinants

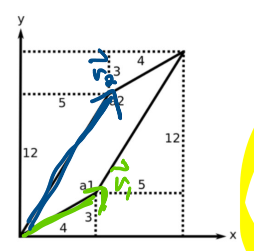
(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

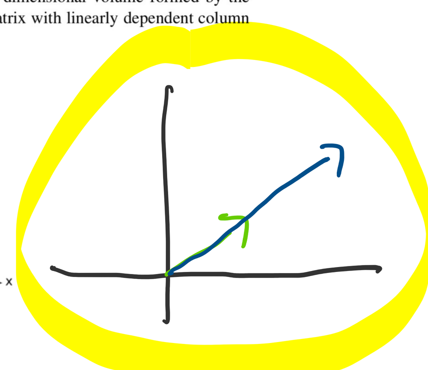
$\text{Det}(A) = ad - bc$
$(2)(3) - 0 = \boxed{6}$

(b) Compute the determinant of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

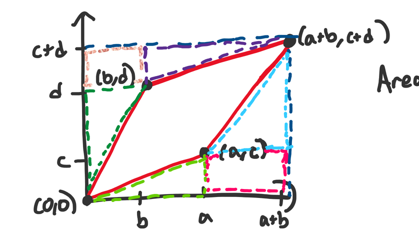
$\text{Det}(A) = (2)(3) - (1)(0) = \boxed{6}$

(c) We know that the determinant of a matrix represents the multi-dimensional volume formed by the column vectors. Explain intuitively why the determinant of a matrix with linearly dependent column vectors is always 0.





Area of parallelogram = Determinant



$$\text{Area} = \underbrace{(a+b)(c+d)}_{\text{area of large rectangle}} - \underbrace{\left(\frac{ac}{2}\right)}_{\text{area of bottom triangle}} - \underbrace{\left(\frac{(a+b-b)(c+d-d)}{2}\right)}_{\text{area of top triangle}} - \underbrace{\left(\frac{bd}{2}\right)}_{\text{area of left triangle}} - \underbrace{\left(\frac{(a+b-a)(c+d-c)}{2}\right)}_{\text{area of right triangle}} - \underbrace{(a+b-a)c}_{\text{area of smaller rectangle at the bottom right}} - \underbrace{b(c+d-c)}_{\text{area of smaller rectangle at top left}}$$

$$= \cancel{ac} + ad + bc + \cancel{bd} - \cancel{\frac{ac}{2}} - \cancel{\frac{ac}{2}} - \cancel{\frac{bd}{2}} - \cancel{\frac{bd}{2}} - 2bc$$

$= \boxed{ad - bc}$ equivalent to determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors. State if the inverse of $\mathbf{M}$ exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

1. $M\vec{x} = \lambda \vec{x}$ ($\vec{x}$: eigenvectors, λ: eigenvalue) $\quad (M - \lambda)x = 0 \quad (M - \lambda I)x = 0$

$\det(M - \lambda I) = \begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$(0-\lambda)(-3-\lambda) - (-2)(1) = \lambda^2 + 3\lambda + 2 = 0 \quad (\lambda+2)(\lambda+1) = 0 \quad \lambda_1 = -2,\ \lambda_2 = -1$

2. $\lambda_1 = -2$: $\begin{bmatrix} 2 & 1 \\ -2 & -3-(-2) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ $\quad B\vec{x} = 0 \quad \vec{x} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Shortcut is to see what linear combination of cols = 0

$\lambda_2 = -1$: $\begin{bmatrix} 0-(-1) & 1 \\ -2 & -3-(-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenspace: span of the eigenvectors for given eigenvalue

$\lambda_1 = -2$ span$\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right\}$ $\quad \lambda_2 = -1$; span$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

1. Find eigenvalues

$\text{Det}\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = 0 \quad (-2-\lambda)(8-\lambda) + 16 = 0 \quad \lambda^2 - 6\lambda = 0 \quad \boxed{\lambda_1 = 0\ \lambda_2 = 6}$

2. Find eigenvectors

$\lambda_1 = 6$: $\begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\lambda_2 = 0$: $\begin{bmatrix} -2 & 4 \\ 4 & 8 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\boxed{\lambda_1 = 6\ v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \lambda_2 = 0\ v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

Any scalar multiple of the eigenvector is an eigenvector

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad (M - \lambda I)\vec{x} = 0$

1. $\text{Det}(M - \lambda I) = 0$

$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0-\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ $\quad \boxed{\lambda_1 = 0 \\ \lambda_2 = 0}$

$\cancel{(0-\lambda) - \lambda}$ $(0-\lambda) \cdot -\lambda - 0 = 0 \quad \boxed{\lambda^2 = 0} \quad \lambda_1 = 0,\ \lambda_2 = 0$

2. $\begin{bmatrix} 0-0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad 0x_1 + x_2 = 0 \quad x_1 = x_1 \quad x_1 = 1,\ x_2 = 0$

$\begin{array}{c} x_1 \\ x_2 \end{array} \boxed{\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$M - \lambda I = M\vec{x} = 0$ ($\lambda = 0$)

eigenspace: $\lambda = 0$ span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ $\quad$ Span$\{x\}$ $\quad c \cdot \vec{x}$

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

for a given e-val, its e-vecs will be LI

Eigenvalue recap:

$M \cdot \vec{x} = \lambda \vec{x}$ (λ: eigenvalue) $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 5$

$M\vec{x} - \lambda x = \vec{0}$

$\underbrace{(M - \lambda I)}_{B} \vec{x} = \vec{0}$

$B\vec{x} = \vec{0}$ $\quad N(B) \neq \{\vec{0}\} \iff \text{Det}(B) = 0$

~~Nut~~ $N(B) = N(M - \lambda I)$

1. $\text{Det}(M - \lambda I) = 0$

$\begin{bmatrix} 5-\lambda & 1 \\ 1 & \lambda \end{bmatrix} \quad (5-\lambda)\lambda - 1 = 0$

2. Find eigenvectors

$\begin{bmatrix} 5-c & 1 \\ 1 & c \end{bmatrix}$

$\left[\begin{array}{cc|c} 5-c & 1 & 0 \\ 1 & c & 0 \end{array}\right]$

$(M - \lambda I) \cdot x = 0$ (x: eigenvectors)

$M\vec{x} = \lambda\vec{x}$

1. $\text{Det}(M - \lambda I) \to$ solve char. polynomial
Gives us eigenvalues

2. Use GE for $M - \lambda I$ ~~on~~ and check what vectors are $\in N(M - \lambda I)$
Eigenvectors

$\lambda = 0$
$\lambda_1 = 1$
$\lambda_2 = 1$ $\Rightarrow$ 2 e-vecs
1 e-vec

(d) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(e) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

$\mathbb{R}^2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \lambda_1 = 1,\ \lambda_2 = 1$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{x} = 1 \cdot \vec{x}$ eigenvectors

$\lambda = 1$: span$\underbrace{\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}_{\text{eigenbasis}}$

$M - \lambda I \quad \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \quad (1-\lambda)(1-\lambda) = 0 \quad \lambda^2 - 2\lambda + 1 = 0 \quad (\lambda-1)(\lambda-1) = 0 \quad \boxed{\lambda_1 = 1 \\ \lambda_2 = 1}$

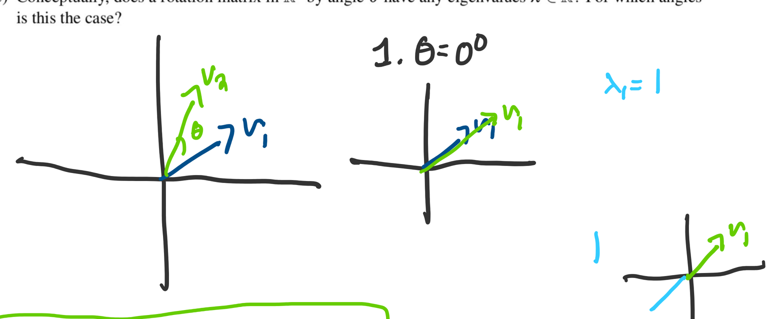
(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \\ 0 & & \ddots & \\ 0 & \cdots & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad$ eigenvalues: d_i's

$\begin{bmatrix} d_1 & 0 & \cdots \\ 0 & d_2 & \\ \vdots & & \ddots \ d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = d_1 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

eigenvalues: d_i's
eigenvectors: d_1 $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ unit vectors where the i^{th} component is 1

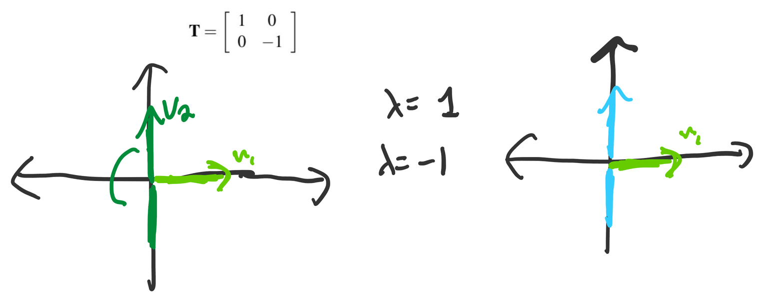
(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?



$\boxed{1.\ \theta = 0^0 \quad \lambda = 1 \\ 2.\ \theta = 180^0 \quad \lambda = -1}$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$Mx = \lambda x$
$\lambda_1 = 1$
$\lambda_1 = -1$

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \lambda = 0 \quad (M - \lambda I)x = 0 \quad \lambda = 0 \quad M \cdot \vec{x} = \vec{0}$

$\boxed{\infty \text{ sol}}$

M has a non-trivial nullspace

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?