EECS 16A Dis 7A Notes

Friday, September 3, 2021 1:38 PM

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1. Proof *Topics: proof, null space, invertibility.* Consider a square matrix **A**. Prove that if **A** has a non-trivial nullspace, i.e. if the nullspace of **A** contains more than just $\vec{0}$, then matrix **A** is not invertible.

$A\vec{x} = \vec{0}$

$\uparrow$

$\vec{x} = \vec{0}$

$A^{-1}(A\vec{y}) = A^{-1}\cdot(\vec{0})$

$\boxed{\vec{y} \neq \vec{0}}$ *

$A^{-1}A = I$

$\vec{y} = \vec{0}\cdot A^{-1} = \vec{0}$

$\boxed{\vec{y} = \vec{0}}$ If A has a non-trivial nullspace, A is not invertible

2. The Romulan Ruse While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

(a) *Concept: Matrix Transformations*

The Romulan illusion technology causes a point (x_0, y_0) to transform or *map* to (u_0, v_0). Similarly, (x_1, y_1) is mapped to (u_1, v_1). Figure 1 and Table 1 show two points on a Romulan ship and the corresponding *mapped* points.

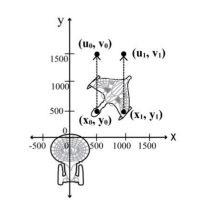


Figure 1: Figure for part (a)

Original Point	Mapped Point
$(x_0, y_0) = (500, 500)$	$(u_0, v_0) = (500, 1500)$

Original Point	Mapped Point
$(x_1, y_1) = (1000, 500)$	$(u_1, v_1) = (1000, 1500)$

Table 1: Original and Mapped Points

Find a transformation matrix A_0 such that

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

$$\underbrace{\begin{bmatrix} 500 \\ 1500 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 500 \\ 500 \end{bmatrix}}_{2\times 1 =} (1) \qquad \begin{bmatrix} 1000 \\ 1500 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1000 \\ 500 \end{bmatrix} (2)$$

1. GE

$500a + 500b = 500$ $\quad$ $1000a + 500b = 1000$

$500c + 500d = 1500$ $\quad$ $1000c + 500d = 1500$

$a = 1 \quad b = 0 \quad c = 0 \quad d = 3$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 3y_0 \end{bmatrix}$$

(b) *Concept: Matrix Transformations*

In this scenario, every point on the Romulan ship (x_m, y_m) is mapped to (u_m, v_m), such that vector $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$ is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

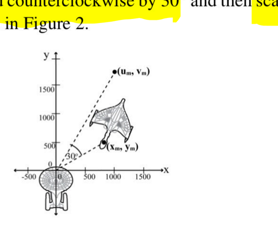


Figure 2: Figure for part (b)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Table 2: Trigononometric Table

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = 2R_\theta$$

Find a transformation matrix R such that $\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \mathbf{R}\begin{bmatrix} x_m \\ y_m \end{bmatrix}$.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}}_{R_\theta} \qquad \boxed{2\cdot R_\theta}$$

rotate CCW by θ°

CW rotation $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point $(0,0)$ towards the probe.

(c) *Concept: Gaussian Elimination, Systems of Equations*

The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix $\mathbf{A}_p$:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

They positioned the probe at (x_p, y_p) so that it maps to $(u_p, v_p) = (0,0)$, where $\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}$. This scenario is shown in Figure 3. The initial position of the torpedo is $(0,0)$ and the torpedo cannot be fired on its initial position! Impressive trick indeed!

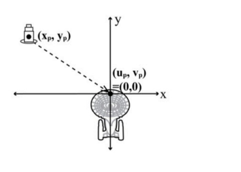


Figure 3: Figure for part (c)

Find the possible positions of the probe (x_p, y_p) so that $(u_p, v_p) = (0,0)$.

$A_p \cdot \vec{p} = \vec{0}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0} \qquad \boxed{\alpha\begin{bmatrix} -3 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

(d) *Concept: Eigenspaces/Eigenvectors/Eigenvalues*

It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, $\mathbf{A}_p$, mapped it to (u_q, v_q), where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo while traveling along a straight line from $(0,0)$ to (u_q, v_q), hit the probe at (x_q, y_q) on the way!

The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

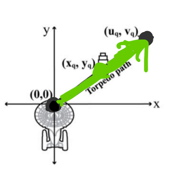


Figure 4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on its initial position $(0,0)$. This means that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$ cannot be $(0,0)$.

$$A_p \cdot \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$$

1. $\det(A - \lambda I) = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}$$

$(1-\lambda)(6-\lambda) - 6 = 0$

$\lambda^2 - 7\lambda + \cancel{6} - \cancel{6} = 0$

$\lambda^2 - 7\lambda = 0 \quad \cancel{\lambda = 0} \quad \lambda = 7$

$A_p \cdot p \neq 0 \quad \lambda \neq 0 \qquad A_p \cdot \vec{p} = \underset{\lambda = 0}{\lambda} \cdot \vec{p} \qquad A_p \cdot \vec{p} = \vec{0} \quad \begin{bmatrix} u_q \\ v_q \end{bmatrix}$

2. Solve for eigenvectors

$$\begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 1-7 & 3 \\ 2 & 6-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix}$$

$(A - \lambda I)\cdot\vec{x} = \vec{0}$

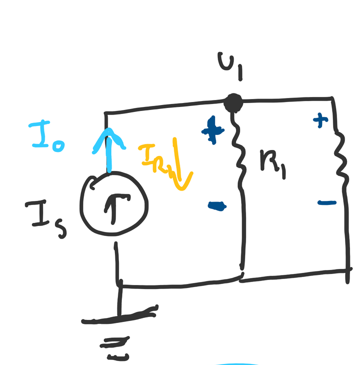
$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

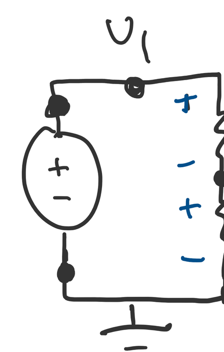
$$1\begin{bmatrix} -6 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boxed{\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}} \qquad \boxed{\alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\underbrace{\alpha\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{all possible positions}}$$

3. Derive I_{R_2} (current going through R_2) in terms of I_s, R_1, and R_2.





$$V_{R_2} = \frac{R_2 \cdot V_s}{R_1 + R_2}$$

$V_{R_1} = u_1 - u_{mid} = V_s - u_{mid}$

$V_s = u_1 - GND$

$V_{R_2} = u_{mid} - GND$

$V_{R_2} = u_{mid}$

$V = IR$

$\uparrow$

$-V = -I \cdot R$

$u_1 = V_s$

KCL: $I_0 = I_{R_1} + I_{R_2}$

$V_{R_1} = u_1 - 0 \qquad V_{R_2} = u_1 - 0$

$V_{R_1} = V_{R_2}$

$V_{R_1} = I_{R_1}\cdot R_1 \qquad V_{R_2} = I_{R_2}\cdot R_2$

$V_{R_1} = V_{R_2}$

$I_{R_1}\cdot R_1 = I_{R_2}\cdot R_2$

$$I_{R_1} = \frac{I_{R_2}\cdot R_2}{R_1}$$

$I_0 = I_s$

$I_0 = I_s = I_{R_1} + I_{R_2}$

$$I_s = \frac{I_{R_2}\cdot R_2}{R_1} + \frac{(I_{R_2})R_1}{R_1}$$

$$I_s = \frac{I_{R_2}(R_1 + R_2)}{R_1}$$

$$\boxed{\frac{I_s \cdot R_1}{R_1 + R_2} = I_{R_2}}$$

$$V_{R_2} = \frac{R_2 \cdot V_s}{R_1 + R_2}$$