EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 2

This homework is due September 10, 2021, at 23:59.
Self-grades are due September 13, 2021, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please review Note 1B and read Note 2A. They will provide an overview of Gaussian elimination, vectors, and matrices. You are always welcome and encouraged to read beyond this as well, in particular, a quick look at Note 3 will help you. Please answer the following questions: How can Gaussian elimination help you determine if there are no solutions to a particular system of equations? How can Gaussian elimination help you determine if a particular system has a unique solution? How about an infinite number of solutions? Does a row of zeros always mean there are infinite solutions?

2. Campfire Smores

Patrick and SpongeBob are making smores.

There are three ingredients: **Graham Crackers, Marshmallows, and Chocolate**. To make a smore, SpongeBob needs: s_g Graham Crackers, s_m number of Marshmallows, and s_c Chocolate.

Ingredients	**Amount Needed**
Graham Crackers (s_g)	17
Marshmallows (s_m)	20
Chocolate (s_c)	24

Table 1: SpongeBob's smore

They find out that these ingredients are only stored in bundles as below:

Lobster Pack (p_l)
8 graham crackers
6 marshmallows
6 chocolates

Mr. Krabs Pack (p_k)
2 graham crackers
2 marshmallows
21 chocolates

Squidward Pack (p_s)
3 graham crackers
3 marshmallows
5 chocolates

Gary Pack (p_g)
1 graham crackers
3 marshmallows
5 chocolates

Pearl Pack (p_p)
2 graham crackers
7 marshmallows
4 chocolates

Table 2: Amount of Ingredients per Bundle

Spongebob and Patrick need to know how many of each bundle to buy: number of "Lobster" Packs, p_l, number of "Mr. Krabs" Packs, p_k, number of "Squidward" Packs, p_s, number of "Gary" Packs, p_g, and number of "Pearl" Packs, p_p.

(a) How many equations/constraints does the information in the problem provide you with?

3. Gaussian Elimination

Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x + 2y + 5z &= 3 \\ x + 12y + 6z &= 1 \\ 2y + z &= 4 \\ 3x + 16y + 16z &= 7 \end{aligned}$$

(c) Consider the following system of equations:

$$x + 2y + 5z = 6$$
$$3x + 9y + 6z = 3$$

You are given a set S of candidate solutions,

$$S = \left\{ \vec{v} \;\middle|\; \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} t \quad , \quad t \in \mathbb{R} \right\}$$

This vector notation can be expressed in terms of its components:

$$\vec{v} = \begin{bmatrix} 16 - 11t \\ -5 + 3t \\ t \end{bmatrix} \quad \text{means} \quad \begin{aligned} x &= 16 - 11t \\ y &= -5 + 3t \\ z &= t \end{aligned}$$

Show, by substitution, that any $\vec{v} \in S$ is a solution to the system of equations given above. Note that this means that the candidate solution must satisfy the system of equations for all $t \in \mathbb{R}$.

(d) Consider the following system:

$$4x + 4y + 4z + w + v = 1$$
$$x + y + 2z + 4w + v = 2$$
$$5x + 5y + 5z + w + v = 0$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

4. Mechanical Linear Algebra

(a) Consider the following system of equations:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = 2 \\ -4x_1 - 6x_2 - 2x_3 = -2 \\ x_2 - 0.2x_3 = 1 \end{cases}$$

Write the system of equations in augmented matrix form and bring to **reduced row echelon form** through Gaussian elimination. **How many solutions (if any) does this system of equations have?**

(b) For the new row reduced matrix below:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & d-4 \end{array}\right]$$

For what value of d (if any) will the system have a solution? If it does have a solution, what is the solution?

5. Word Problems

Learning Objective: *Understand how to setup a system of equations from word problems and solve using Gaussian elimination.*

For these word problems, represent the system of equations as an augmented matrix and solve. **We explicitly want you to use Gaussian elimination to solve.**

(a) Gustav is collecting soil samples. Each soil sample contains some sand, some clay, and some organic material. He wants to know the density of each material. His first sample has 0.5 liters of sand, 0.25 liters of clay, and 0.25 liters of organic material, and weighs 1.625 kg. His second sample contains 1 liter of sand, 0 liters of clay, and 1 liter of organic material, and weighs 3 kg. His third sample contains 0.25 liters of sand, 0.5 liters of clay, and 0 liters of organic material, and weighs 1.375 kg. That is,

$$0.5s + 0.25c + 0.25m = 1.625 \tag{4}$$

$$1s + 0c + 1m = 3 \tag{5}$$

$$0.25s + 0.5c + 0m = 1.375 \tag{6}$$

where s is the density of sand, c is the density of clay, and m is the density of organic material, all measured in kg/L. Solve for the density of each material.

(b) Alice buys 3 apples and 4 oranges for 17 dollars. Bob buys 1 apple and 10 oranges for 23 dollars (Bob really likes oranges). How much do apples and oranges cost individually?

(c) Jack, Jill, and James are driving from Berkeley to Las Vegas. Each of them takes a different route. Jack takes a short route and ends up going through Toll Road A and Toll Road B, costing him \$10. Jill takes a slightly longer route and goes through Toll Road B and Toll Road C, costing her \$15. Finally, James takes a wrong turn and takes Toll Road A twice, then takes Toll Road B and finally Toll Road C, costing him \$25. What is the toll cost on each road?

6. Ball weights

Learning Objective: *Understand how to setup a system of equations from word problems.*

For these word problems, you only need to setup the problem with Gaussian elimination or matrix-vector notation. Of course, you may solve for practice, but no additional credit is awarded.

Your company wants to build a machine to automatically weigh and sort bowling balls. Your boss has asked you to build a demonstration using just 4 balls. She has given you a scale and 4 colored balls: blue, green, red, and black. These balls have weights w_b, w_g, w_r, and w_k respectively. Your goal is to determine the weight of each ball.

The simplest way you can determine the ball weights is to weigh them one at at time. That is, you will first weigh only the blue ball, then only the green ball, etc, in the order given above. After 4 measurements, the scale outputs 4 values corresponding to each measurement: y_1, y_2, y_3, and y_4. The augmented matrix setup would look like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & y_3 \\ 0 & 0 & 0 & 1 & y_4 \end{array}\right]$$

Alternatively, the matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_b \\ w_g \\ w_r \\ w_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(a) As you manually put the balls on the scale, you make a mistake and accidentally measure the balls in this order: red, black, blue, green. The scale outputs 4 values: a_1, a_2, a_3, a_4. Keeping the vector order the same as above (w_b, w_g, w_r, w_k), what would the augmented matrix look like?

(b) Your boss has now given you a robotic arm to pick up balls and put them on the scale. While setting up the code, you forget to tell the robot to remove the previous ball(s) already on the scale from the previous measurement. As a result, measurement one contains the blue ball, measurement two contains the blue ball and the green ball, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. Setup the matrix problem for this.

(c) While working from home, your cat accidentally jumps on your keyboard, causing a new code mistake. Now, during each measurement, instead of the robot putting one ball on the scale, it places the other 3 balls on the scale. For example, instead of measuring just the blue ball, the robot measures the green, red, and black balls together. The output of the 4 measurements are w_1, w_2, w_3, and w_4. Setup the matrix problem for this.

7. Vector-Vector Multiplication

Learning Objective: *Practice evaluating matrix-vector multiplication.*

For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

(a) i. $\vec{x}^T \cdot \vec{z}$

ii. $\vec{x} \cdot \vec{x}^T$

iii. $\vec{x} \cdot \vec{y}^T$

iv. $\vec{x} \cdot \vec{z}^T$

(b) Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

i. Measure and draw a coordinate plane, and carefully plot $\vec{x}$ and $\vec{y}$.

ii. Compute the vector-vector product $\vec{y}^T \vec{x}$.

iii. Using a ruler or other measuring tool, measure the length of $\vec{x}$, denoted l, and the length u of one unit on your coordinate plane. What is the ratio $\frac{l}{u}$? (Note: This ratio is just the length of vector $\vec{x}$ adjusted for the size of your drawing.)

iv. Compute $\sqrt{\vec{x}^T \vec{x}}$.

v. Comment on your findings in parts (iii) and (iv).

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.