EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 3

This homework is due September 17, 2021, at 23:59.
Self-grades are due September 20, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw3.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit each file to its respective assignment on Gradescope.

1. **Reading Assignment** For this homework, please read Notes 2B-4. Notes 3 and 4 will give you an overview of linear independence, span and an introduction to thinking about and writing proofs. Please write a few sentences about how you can use the strategies in the notes to tackle proof questions.

2. **Filtering Out The Troll**

   Learning Goal: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

   You attended a very important public speech and recorded it using two directional microphones (one microphone receives sound from the x direction and the other from the y direction). However, someone in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

   The diagram shown in Figure 1 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

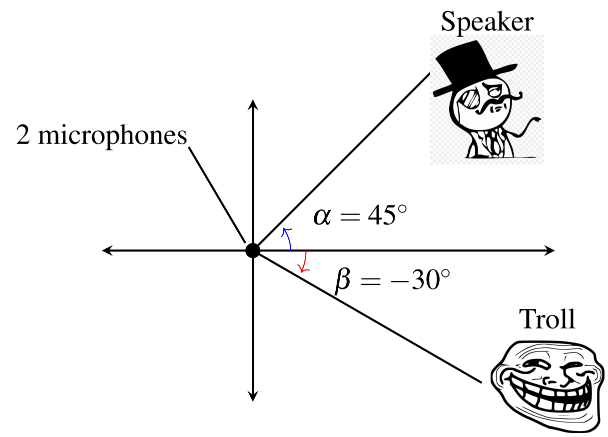

Figure 1: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis. The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha)\cdot\vec{s} + \cos(\beta)\cdot\vec{r} \tag{1}$$

$$\vec{m}_2 = \sin(\alpha)\cdot\vec{s} + \sin(\beta)\cdot\vec{r} \tag{2}$$

where α and β are the angles at which the public speaker and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the public speaker and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 1 and 2 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

(b) Solve the system using any convenient method you prefer from the earlier part to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c\cdot\vec{m}_1 + k\cdot\vec{m}_2$ (where c and k are scalars). What are the values of c and k?

(c) Partial IPython code can be found in `prob3.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

3. Multiply the Matrices

Learning Objective: *Practice evaluating matrix-matrix multiplication.*

(a) We have two matrices $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ is a 3×2 matrix and $\mathbf{B}$ is a 2×4 matrix. Would the multiplication $\mathbf{AB}$ be a valid operation? If yes, what do you expect the dimensions of $\mathbf{AB}$ to be?

(b) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

(c) Now let us assume $\mathbf{A} \in \mathbb{R}^{2 \times n}$ is a new matrix with 2 rows, which are given by the transposes of column vectors $\vec{r}_1$, $\vec{r}_2$ i.e.

$$\mathbf{A} = \begin{bmatrix} - & \vec{r}_1^{\,T} & - \\ - & \vec{r}_2^{\,T} & - \end{bmatrix} \qquad \text{where,} \qquad \vec{r}_1 = \begin{bmatrix} r_{11} \\ r_{12} \\ \vdots \\ r_{1n} \end{bmatrix}, \text{ and } \qquad \vec{r}_2 = \begin{bmatrix} r_{21} \\ r_{22} \\ \vdots \\ r_{2n} \end{bmatrix}.$$

$\mathbf{B} \in \mathbb{R}^{n \times 3}$ is a new matrix with 3 columns, which are called $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$, i.e.

$$\mathbf{B} = \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \qquad \text{where,} \qquad \vec{c}_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \end{bmatrix}, \qquad \vec{c}_2 = \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \end{bmatrix}, \text{ and } \qquad \vec{c}_3 = \begin{bmatrix} c_{31} \\ c_{32} \\ \vdots \\ c_{3n} \end{bmatrix}$$

Now show that:

$$\mathbf{AB} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{c}_1 & \vec{r}_1^{\,T}\vec{c}_2 & \vec{r}_1^{\,T}\vec{c}_3 \\ \vec{r}_2^{\,T}\vec{c}_1 & \vec{r}_2^{\,T}\vec{c}_2 & \vec{r}_2^{\,T}\vec{c}_3 \end{bmatrix}$$

if $\mathbf{AB}$ is a valid operation.

4. Linear Dependence

Learning Goal: *Evaluate the linear dependency of a set of vectors.*

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

5. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

6. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 2 shows how the

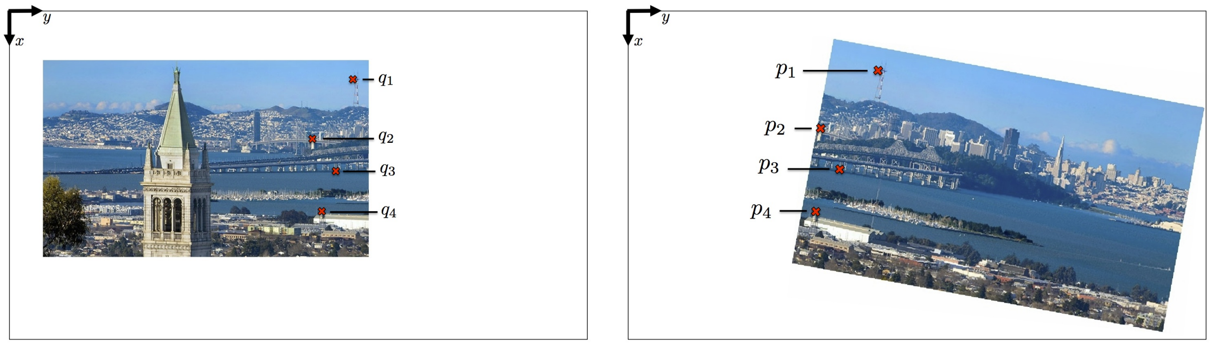

Figure 2: Two images to be stitched together with pairs of matching points labeled.

points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{3}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this equation similar to Equation (3),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{4}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob3.ipynb`.
Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation

(ii) Scaling

(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob3.ipynb`.

(b) Multiply Equation (3) out into **two linear equations**.

(i) What are the known values and what are the unknowns in each equation?

(ii) How many unknowns are there?

(iii) How many independent equations do you need to solve for all the unknowns?

(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint:*** *Remember that each pair of $\vec{p}$ and $\vec{q}$ will give you* ***two*** *different linear equations.*

(c) Write out a system of linear equations that you can use to solve for $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$. Assume that all four pairs of points from Fig. 2 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Now think of your answer to Part b(iv). How many pairs of these points would you need to solve for $\vec{\alpha}$. Choose **just enough** equations required to solve for $\vec{\alpha}$ and express these linear equations using **matrix-vector form**.

(d) In the IPython notebook `prob3.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, $\mathbf{R}$, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.**

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.