# EECS 16A Designing Information Devices and Systems I
# Fall 2021 Homework 4

**This homework is due September 24, 2021, at 23:59.**
**Self-grades are due September 27, 2021, at 23:59.**

**Submission Format**

Your homework submission should consist of **one** file.

`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit each file to its respective assignment on Gradescope.

**1. Reading Assignment**

For this homework, please review Note 5 and read Notes 6, 7. The notes 5 and 6 provide an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps, and matrix inversion. Note 7 provides an introduction to vector spaces. You are always welcome and encouraged to read beyond this as well. Note 8 discusses column spaces and nullspaces, so it might be useful to read that for this homework as well.

You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

**2. Feedback on your study groups**

Please help us understand how your study groups are going! Fill out the following survey to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

https://forms.gle/xhPwbFZzMWN1dBrn7

To get full credit for this question you must (1) fill out the survey (it will record your email) and (2) indicate in your homework submission that you filled out the survey.

**3. Easing into Proofs**

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***Learning Objectives:*** *This is an opportunity to practice your proof development skills.*

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of A are linearly dependent.** Let us use the structure delineated in **Note 4** to approach this proof. This problem has 4 sub-parts and the following is a chart showing the sequential steps we are going to take to approach this proof.

In a text book you might see the steps in a proof written out in the order in the middle column of the table. But when you are building a proof you usually want to go in another order — this is the order of the subparts in this problem.

| Proof steps | | Corresponding problem sub-parts |
|---|---|---|
| 1 | Write what is known | Sub-part (i) |
| 2 | Manipulate what is known | Sub-part (iii) |
| 3 | Connecting it up | Sub-part (iv) |
| 4 | What is to be shown | Sub-part (ii) |

(i) **First Step: write what you know**
Think about the *information we already know* from the problem statement. We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions. Infinitely many solutions are hard to work with, but perhaps we can simplify to something that we can work with. If the system has infinite number of solutions, it must have at least ___ distinct solutions (Fill in the blank).
So let us assume that $\vec{u}$ and $\vec{v}$ are two different vectors, both of which are solutions to $\mathbf{A}\vec{x} = \vec{b}$.
Express the sentence above in a mathematical form (Just writing the equations will suffice; no need to take do further mathematical manipulation).

(ii) **What we want to show:**
Now consider *what we need to show*. We have to show that the columns of $\mathbf{A}$ are linearly dependent. Let us assume that $\mathbf{A}$ has columns $\vec{c_1}$, $\vec{c_2}$, ..., and $\vec{c_n}$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c_1} & \vec{c_2} & \cdots & \vec{c_n} \\ | & | & \cdots & | \end{bmatrix}$. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of $\vec{c_1}$, $\vec{c_2}$, ..., and $\vec{c_n}$.

(iii) **Manipulating what we know:**
Now let us try to start from the **First step: equations from (i)**, make mathematically logical steps and reach the **What we want to show: equations from (ii)**. Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the the column vectors too. For example, we can write

$$\mathbf{A}\vec{x} = \vec{b}$$

$$\implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c_1} & \vec{c_2} & \cdots & \vec{c_n} \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \vec{b}$$

$$\implies x_1\vec{c_1} + x_2\vec{c_2} + \ldots + x_n\vec{c_n} = \vec{b}$$

Notice that $x_1$,... $x_n$ etc are scalars. Now use your answer to part (i) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$. Note that this is proceeding slightly differently from how we did this proof in lecture. This is fine — there are often many correct ways to do a proof.

(iv) **Connecting it up:**
Now think about how you can mathematically manipulate your answer from part (iii) (**Manipulating what we know**) to **match the pattern** of your answer from part (ii) (**What we want to show**).

(b) **[PRACTICE]** Now try this proof on your own. Similar proofs will also be covered in your discussion section 3A. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In order to show this, you have to proof the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.

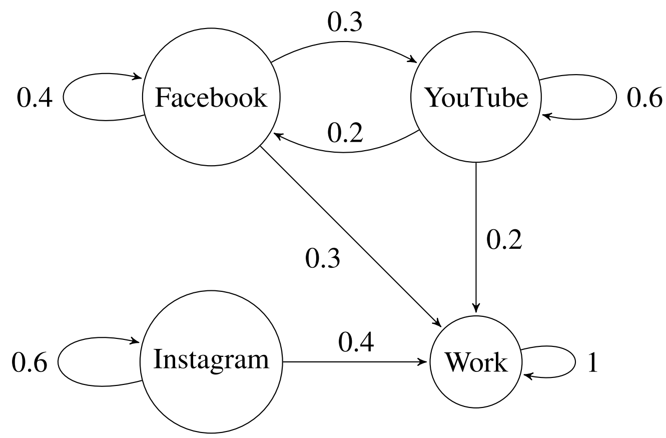
- If a vector $\vec{r}$ belong is span$\{\vec{v}_1+\vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to proof the problem statement from both directions. Now use the method developed in part (a) to proof these statements.

## 4. Social Media

***Learning Objective:*** *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.




(a) Let us define $x_F[n]$ as the number of students on Facebook at timestep $n$, $x_Y[n]$ as the number of students on YouTube at timestep $n$, $x_I[n]$ as the number of students on Instagram at timestep $n$, and $x_W[n]$ as the number of students working at timestep $n$. Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_F[n] \\ x_Y[n] \\ x_I[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix.

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Facebook, 450 on YouTube, 200 on Instagram, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

## 5. Mechanical Inverses

***Learning Objectives***: *Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

**For each of the following values of matrix A:**

i Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii **For parts (a)-(d)**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the $x$ axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii **For parts (a)-(d)**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $\cos\theta \neq 0$. *Hint:* $\cos^2\theta + \sin^2\theta = 1$.

(e) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

(f) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

(g) (**OPTIONAL**) $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(i) **(OPTIONAL)**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

*Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!*

## 6. Finding Null Spaces and Column Spaces

***Learning Objectives**: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

**Definition (Null space)**: The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $\mathrm{N}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{N}(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

**Definition (Column space)**: The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as $\mathrm{C}(\mathbf{A})$ or $\mathrm{range}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{C}(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

**Definition (Dimension)**: The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $\mathrm{C}(\mathbf{A})$?

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{A})$ (i.e. a basis for $\mathrm{C}(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $\mathrm{C}(\mathbf{A})$?

*Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.*

(c) Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{A})$ (i.e. a basis for $\mathrm{N}(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $\mathrm{N}(\mathbf{A})$?

(d) Find the sum of the dimensions of $\mathrm{N}(\mathbf{A})$ and $\mathrm{C}(\mathbf{A})$. What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{B})$ (i.e. a basis for $\mathrm{C}(\mathbf{B})$). What is the minimum number of vectors required to span the $\mathrm{C}(\mathbf{B})$?

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{G})$, i.e. a basis for $\mathrm{N}(\mathbf{G})$.

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

(g) (**OPTIONAL**) For the following matrix $\mathbf{D}$, find $\mathrm{C}(\mathbf{D})$ and its dimension, and $\mathrm{N}(\mathbf{D})$ and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

**7. Cubic Polynomials**

***Learning Goal:*** *This problem shows us that we can treat fixed-degree polynomials as a vector space. Furthermore, many operations on polynomials are linear operations in this vector space and can be represented by matrices.*

(a) Show that the set of all cubic polynomials

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3,$$

where $t \in [a,b]$ and the coefficients $p_k$ are real scalars, forms a vector space. Call this vector space V.

(b) Consider the set of real-valued monomials given below:

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \varphi_2(t) = t^2, \quad \varphi_3(t) = t^3,$$

where $t \in \mathbb{R}$.

Show that every real-valued cubic polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

defined over the interval $[a,b]$ can be written as a linear combination of the monomials $\varphi_0(t)$, $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_3(t)$. In particular, show that

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

where

$$\vec{c}^T = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}$$

is a vector of appropriately chosen coefficients and

$$\vec{\varphi}(t) = \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}.$$

(c) The monomials $\varphi_k(t) = t^k$, for $k = 0, 1, 2, 3$, constitute a basis for the vector space of real-valued cubic polynomials defined over the interval $[a,b]$. Justify why this is true. What is the dimension of V?

(d) Express the derivatives of the basis polynomials $\varphi_i(t)$ for $i = 0, 1, 2, 3$ in terms of the $\varphi_i(t)$ for $i = 0, 1, 2, 3$.

(e) Let **D** be a 4x4 matrix. Use the previous part to help you find the entries of **D**, such that for any polynomial

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

its derivative can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = (D\vec{c})^T \vec{\varphi}(t).$$

*Hint:* What are the dimensions of $(\mathbf{D}\vec{c})^T$? (Reminder: The dimensions of a matrix or vector is a different concept than the dimensions of a vector space).

**8. Homework Process and Study Group**

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.