EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 12

This homework is due November 19, 2021, at 23:59.
Self-grades are due November 22, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw12.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

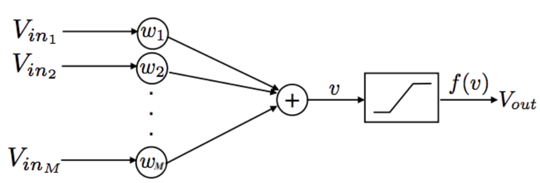
Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please review Note 20 (Op-Amp Current Source and Circuit Design), and read Note 21 (Inner Products and GPS). You are always encouraged to read beyond this as well.

2. Brain-on-a-Chip with 16A Neurons

Neurelic, Inc. is a hot new startup building chips that emulate brain functions like associative memory. As an intern fresh out of 16A, you've been assigned to implement the neural network circuits on this chip. The neural network consists of neurons, each of which can be represented by the block diagram below.

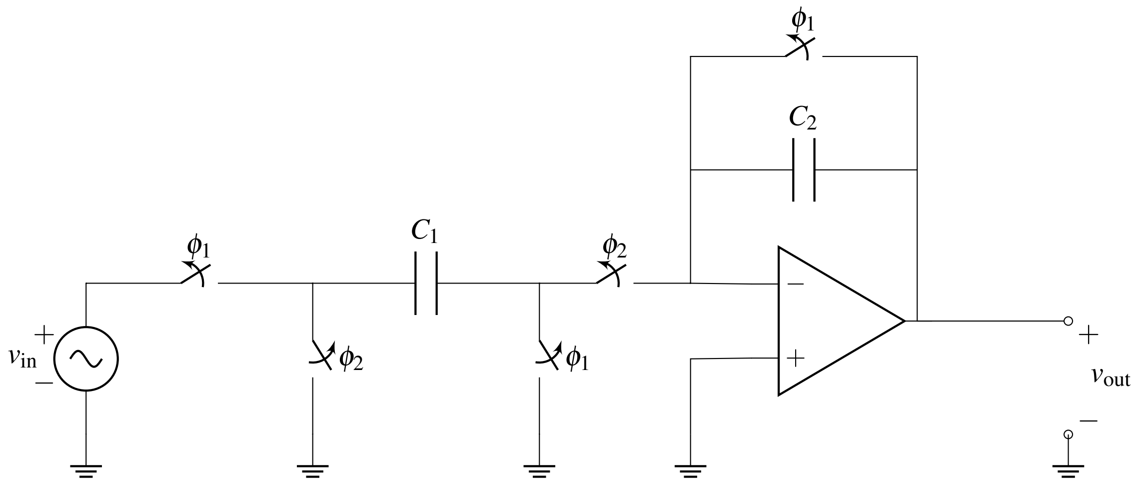


Input signals v_{in_i} are voltages from other neurons, which are multiplied by a constant weight w_i in each synapse and summed in the neuron. Each neuron also contains a nonlinear function (called a sigmoid) which is defined as

$$f(v) = \begin{cases} -1, & v \leq -1 \\ v, & -1 < v < 1 \\ +1, & v \geq +1 \end{cases}$$

where v is the internal neuron voltage after the synapse summation and $f(v)$ is the neuron voltage output.

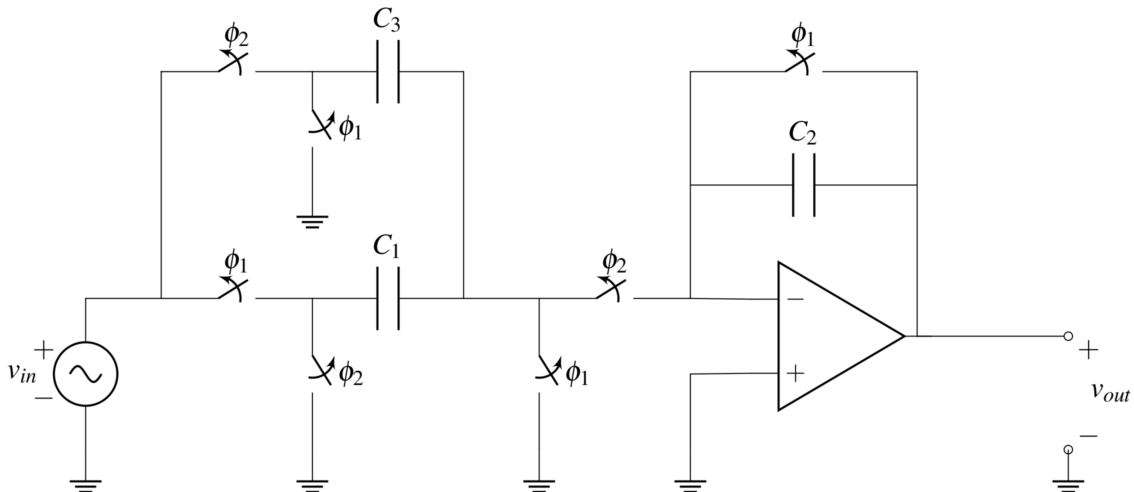
(a) Your mentor suggests that you warm up first by analyzing the circuit below to use as a neuron with a single synapse. ϕ_1 and ϕ_2 are non-overlapping clock phases that control the circuit switches.



i. Draw an equivalent circuit for ϕ_1 and write an expression for v_{out} as a function of v_{in}, C_1, and C_2.
ii. Draw an equivalent circuit for ϕ_2 and write an expression for v_{out} as a function of v_{in}, C_1, and C_2.

(b) Write an equation for v_{out} during ϕ_2 as a function of v_{in} for $C_1 = C_2$ and op-amp supply voltages of ± 1 V. Briefly explain how this circuit implements the sigmoid function.

(c) Then, your mentor shows you the following neuron circuit, which can realize both positive and negative synapse weight and create $v_{out} = w_1 v_{in}$ in ϕ_2.



i. Draw an equivalent circuit during ϕ_1 and write an expression for v_{out} as a function of v_{in}, C_1, C_2, and C_3.
ii. Draw an equivalent circuit during ϕ_2 and write an expression for v_{out} as a function of v_{in}, C_1, C_2, and C_3.

(d) **(Optional)** Now it is your turn to implement a neuron that realizes the function $v_{out} = w_1 v_{in_1} + w_2 v_{in_2}$. Draw the circuit, such that $w_1 = 1/2$ and $w_2 = -1/4$. Label all circuit elements appropriately. You should use a single op-amp and as many capacitors and switches as you need. All capacitors must be of capacitance C_{unit}. Assume that the op-amp power supplies are $\pm$1V (no need to draw them in the circuit). The circuit should operate in 2 phases, resetting in the first phase ϕ_1 and setting $v_{out} = w_1 v_{in_1} + w_2 v_{in_2}$ in the second phase ϕ_2.

3. Inner Product Properties

Learning Goal: *The objective of this problem is to exercise useful identities for inner products.*

Our definition of the inner product in $\mathbb{R}^n$ is:

$$\langle \vec{x}, \vec{y} \rangle \ = \ x_1 y_1 + x_2 y_2 + \ldots + x_n y_v \ = \vec{x}^{\mathsf{T}} \vec{y} \ , \quad \text{for any } \ \vec{x}, \vec{y} \in \mathbb{R}^n$$

Prove the following identities in $\mathbb{R}^n$:

(a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

(c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

(d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(e) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

4. Inner Products

For each of the following functions, show whether it defines an inner product on the given vector space. If not, give a counterexample.

(a) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q}$$

(b) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \vec{q}$$

(c) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{q}$$

(d) For $\mathbb{P}_2$ (the vector space containing all polynomials of up to degree 2):

$$\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

5. Cauchy-Schwarz Inequality

Learning Goal: *The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.*

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, where $r > 0$, $t > 0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r, t > 0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$? *Hint: Recall the trig identity:* $\cos^2 x + \sin^2 x = 1$

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle \vec{v}, \vec{w} \rangle$? *Hint: The trig identity* $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ *may be useful.*

(c) Show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$. *Hint: consider your results from part (b). Also recall* $-1 \leq \cos x \leq 1$ *and use both inequalities.*

(d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

6. Orthogonal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthogonal matrix if $U^{-1} = U^T$.

Orthogonal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthogonal matrices.

(a) Let U be an orthogonal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

(b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.